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APPROXIMATIONS TO THE DISTRIBUTIONS OF THE LIKELIHOOD RATIO STATISTICS FOR TESTING CERTAIN STRUCTURES ON THE COVARIANCE MATRICES OF REAL MULTIVARIATE NORMAL POPULATIONS

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Technology, Incorporated

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In this report, the authors approximated the distributions of the likelihood ratio statistics for testing certain hypotheses on the covariance matrices of the multivariate normal populations by using suitable Pearson type distributions. The hypotheses that are considered include the hypotheses of sphericity, multiple homogeneity of covariance matrices and multiple independence. Using these approximations, tables of percentage points of the distributions considered are constructed. The accuracy of these tables is sufficient for practical purposes.

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PREFACE

This report was prepared for the Applied Mathematics Research Laboratory, Aerospace Research Laboratories, by J. C. Lee, T. C. Chang and P. R. Krishnaiah under Project 7071, "Research in Applied Mathematics", Work Unit 12, "Multivariate Analysis and Its Applications". The work of J. C. Lee and T. C. Chang was performed at the Aerospace Research Laboratories in the capacity of Technology Incorporated Visiting Research Associates under Contract F33615-73-C-4155. J. C. Lee and T. C. Chang are currently on leave of absence from Wright State University and University of Cincinnati, respectively.

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SECTION I

INTRODUCTION

The problems of testing the hypotheses on the structures of the covariance matrix of the multivariate normal population have received considerable attention in the literature since these problems have applications in various disciplines. Wilks [1,2] is one of the earliest workers in this area. In this report, we investigate approximations to the distributions of the likelihood ratio statistics for testing the hypotheses of (1) multiple independence of several sets of variables, (2) sphericity, (3) equality of the covariance matrix to a given matrix, and (4) equality of the covariance matrices of independent sets of variables. These approximations are based upon fitting suitable Pearson type distributions by using the first four moments of the above statistics. Using the above approximations, we construct percentage points of these statistics. The accuracy of these approximations is sufficient for practical purposes.

Exact distributions of the test statistics considered here are very difficult to compute. One can, of course, use Box's asymptotic expression by taking a sufficient number of terms to compute the tables and this may be preferable to exact expressions from a computational point of view. However, this asymptotic expression is also difficult to compute when we need to take several terms in the series and this is the case when the sample size is not large. Since Pearson type approximations are quite simple from a computational point of view and since their accuracy is sufficient for practical purposes, they are definitely preferable to either exact expressions or Box's asymptotic series if we

are interested in computing the percentage points of the statistics considered in this paper.

SECTION II

STATEMENT OF THE PROBLEMS AND PRELIMINARIES

Let $\underset{\sim}{X}' = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_q')$ be distributed as a multivariate normal with mean vector $\underset{\sim}{\mu}'$ and covariance matrix Σ. Also, let $E(\underset{\sim}{X}_i) = \underset{\sim}{\mu}_i$ and $E\{(\underset{\sim}{X}_i - \underset{\sim}{\mu}_i)(\underset{\sim}{X}_j - \underset{\sim}{\mu}_j)'\} = \Sigma_{ij}$, where $\underset{\sim}{X}_i$ is of order $p_i \times 1$ and $s = \sum_{i=1}^{q} p_i$. In this paper, we consider approximations to the distributions of the likelihood ratio statistics for testing the hypotheses H_1, H_2, H_3, H_4, and H_5 where

$$H_1 : \Sigma_{ij} = 0 \quad (i \neq j = 1, \ldots, q)$$

$$H_2 : \Sigma = \sigma^2 \Sigma_o \quad (\sigma^2 \text{ is unknown, } \Sigma_o \text{ is known})$$

$$H_3 : \Sigma = \Sigma_o$$

$$H_4 : \Sigma_{11} = \ldots = \Sigma_{qq} \quad (\text{under the assumption that } H_1 \text{ is true and } p_1 = \ldots = p_q)$$

$$H_5 : \left\{ \begin{array}{l} \Sigma_{11} = \ldots = \Sigma_{q_1,q_1} \\ \Sigma_{q_1+1,q_1+1} = \ldots = \Sigma_{q_2^*,q_2^*} \\ \Sigma_{q_{k-1}^*+1,q_{k-1}^*+1} = \ldots = \Sigma_{q,q} \end{array} \right.$$

Here $q_o^* = 0$, $q_j^* = \sum_{i=1}^{j} q_i$ and $q_k^* = q$.

These distributions are approximated with Pearson type distributions. A brief description of the family of Pearson type distributions is given below.

If $f(x)$, $(c < x < d)$, is the probability density function belonging to the family of Pearson type distributions, then $f(x)$ satisfies the differential equation

$$(b_0 + b_1 x + b_2 x^2) \frac{df(x)}{dx} = (x + a) f(x). \qquad (1)$$

Let $m'_k = \int_c^d x^k f(x)dx$ and $m_k = \int_c^d (x - m'_1)^k f(x)dx$. It is known that

$$\begin{aligned} a &= m_3(m_4 + 3m_2^2)/A \\ b_0 &= -m_2(4m_2m_4 - 3m_3^2)/A \\ b_1 &= -m_3(m_4 + 3m_2^2)/A \\ b_2 &= -(2m_2m_4 - 3m_3^2 - 6m_2^3)/A \\ A &= 10m_2m_4 - 18m_2^3 - 12m_3^2 \end{aligned} \qquad (2)$$

The family of Pearson type distributions is completely specified by its first four moments and the type of the distribution can be determined by examining the quantity κ where

$$\kappa = \frac{\beta_1(\beta_2 + 3)^2}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)}, \qquad (3)$$

$\beta_1 = m_3^2/m_2^3$ and $\beta_2 = m_4/m_2^2$. If κ is negative, we classify it as the Pearson's Type I distribution. When κ is positive, it is classified as the Pearson's Type IV or Type VI accordingly as κ is less than or greater than unity, respectively. For a discussion of Pearson type distributions, see [3]. Tables for the percentage points of Pearson type distributions are given in Johnson, Nixon and Amos [4] for some values of the parameters.

SECTION III

MULTIPLE INDEPENDENCE OF SEVERAL SETS OF VARIABLES

Let $\underset{\sim}{X}'_j = (\underset{\sim}{X}'_{1j},\ldots,\underset{\sim}{X}'_{qj}),(j=1,\ldots,N)$, be jth independent observation on $\underset{\sim}{X}'$ and let

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1q} \\ A_{21} & \cdots & A_{2q} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A_{q1} & \cdots & A_{qq} \end{bmatrix}$$

where

$$A_{\ell m} = \sum_{j=1}^{N} (\underset{\sim}{X}_{\ell j} - \overline{\underset{\sim}{X}}_{\ell\cdot})(\underset{\sim}{X}_{mj} - \overline{\underset{\sim}{X}}_{m\cdot})'$$

and $\overline{\underset{\sim}{X}}_{\ell\cdot} = \frac{1}{N}\sum_{j=1}^{N} \underset{\sim}{X}_{\ell j}$. Then, the likelihood ratio statistic for testing H_1 is known to be

$$V_1 = \frac{|A|}{\prod_{j=1}^{q} |A_{jj}|} \tag{4}$$

The moments of the statistic V_1 are given by

$$E(V_1^h) = \frac{\prod_{i=1}^{s} \Gamma\left[\frac{1}{2}(n+1-i) + h\right] \prod_{i=1}^{q} \left\{ \prod_{j=1}^{p_i} \Gamma\left[\frac{1}{2}(n+1-j)\right] \right\}}{\prod_{i=1}^{s} \Gamma\left[\frac{1}{2}(n+1-i)\right] \prod_{i=1}^{q} \left\{ \prod_{j=1}^{p_i} \Gamma\left[\frac{1}{2}(n+1-j) + h\right] \right\}} \tag{5}$$

where $n = N-1$ and $\Gamma(n)$ is the complete gamma function. The statistic V_1 and its moments were derived by Wilks [2]. According to the likelihood ratio test, we accept or reject H_1 accordingly as $V_1^* \lessgtr c$, where $V_1^* = -2\log V_1$ and

$$P[V_1^* \le c_1 | H_1] = (1 - \alpha) \tag{6}$$

The distribution of V_1 is quite skew. Hence, we approximate the distribution of $V_1^{1/b}$ with a suitable Pearson type distribution, where b is a properly chosen integer. The type of Pearson distribution that has to be fitted is determined by computing κ given by Eq. (3) after replacing m_1', m_2', m_3', and m_4' with μ_{b1}', μ_{b2}', μ_{b3}' and μ_{b4}', respectively, where μ_{bh}' is the h-th moment of $V_1^{1/b}$. By the above method, it is found that the Pearson's Type I curve fits the distribution of $V_1^{1/b}$. The density of the Pearson's Type I distribution is given by

$$f(x) = \frac{1}{\beta(\alpha+1,\ \varepsilon+1)(\sigma-\sigma_o)^{\alpha+\varepsilon+1}}(x-\sigma_o)^{\alpha}(\sigma-x)^{\varepsilon} \tag{7}$$
$$\sigma_o \le x \le \sigma$$

where σ_o, α, σ and ε depend upon a^o, b_o^o, b_1^o and b_2^o. The parameters a^o, b_o^o, b_1^o and b_2^o are respectively equivalent to a, b_o, b_1, and b_2 after replacing m_i' with μ_{bi}'. In this report we consider the case where $p_i = p$.

Box [5] derived asymptotic expressions for a class of likelihood ratio test statistics in multivariate statistical analysis. This class includes the statistics associated with testing the hypotheses H_1, H_2, H_4, and H_5. The number of terms given by Box is not sufficient to get

the desired degree of accuracy in several practical situations. So, the authors gave terms up to $O(n^{-15})$ in the appendix applying the method of Box.

Consul [6] and Mathai and Rathie [7] derived exact expressions for the distribution of V_1, and these expressions are very diffucult to compute. So, the authors studied approximations to these distributions with suitable Pearson type distributions.

Table I gives a comparison of values obtained by the Pearson type approximation and the asymptotic expression of order n^{-13}. In Table I, α_1 is the value of α if we use the Pearson type approximation whereas α_2 is the value of α when we use the asymptotic expression of order n^{-13}, where α is given by

$$p[V_1^* \leq c_1 | H_1] = (1-\alpha). \tag{8}$$

Davis and Field [8] computed the percentage points of $-2\rho \log V_1$ for some values of the parameters by using the Cornish-Fisher type inversion (see Davis [9]) of Box's asymptotic series when

$$1-\rho = \left\{ 2\left(s^3 - \sum_{i=1}^{q} p_i^3\right) + 9\left(s^2 - \sum_{i=1}^{q} p_i^2\right)\right\} \Big/ 6N\left(s^2 - \sum_{i=1}^{q} p_i^2\right)$$

In Table II, the entries under the columns L-C-K are the values of c_1 obtained by the authors with the Pearson type approximation whereas the entries under the columns D-F are the corresponding values obtained by Davis and Field [8].

Tables I and II indicate that the accuracy of the Pearson type approximation to the distribution of V_1 is sufficient for practical purposes. Hence, using the Pearson type approximation, we computed the values of c_1 for $p = 1, 2, 3$; $q = 3, 4, 5$; $\alpha = 0.01, 0.05, 0.10$; and $M = 1(1)20(2)30$, where $M = n - s - 3$. These values are given in Table III. When $q = 2$, it is found that the Pearson type approximation is quite satisfactory, and the results are reported in a companion paper [10] by the authors.

TABLE I

COMPARISON OF THE PEARSON TYPE APPROXIMATION

WITH THE ASYMPTOTIC EXPANSION OF ORDER n^{-13}

FOR THE DISTRIBUTION OF V_1^*

	q=3		p=1	q=5		p=1
n	c_1	α_1	α_2	c_1	α_1	α_2
10	1.913	0.05	0.0499	4.978	0.05	0.0488
10	2.778	0.01	0.0100	6.333	0.01	0.0095
15	1.187	0.05	0.0500	2.947	0.05	0.0497
15	1.723	0.01	0.0100	3.742	0.01	0.0099
20	0.860	0.05	0.0500	2.099	0.05	0.0499
20	1.249	0.01	0.0100	2.663	0.01	0.0100
30	0.555	0.05	0.0500	1.333	0.05	0.0500
30	0.805	0.01	0.0100	1.690	0.01	0.0100

TABLE II

COMPARISON OF THE PEARSON TYPE APPROXIMATION WITH INVERSION OF THE ASYMPTOTIC EXPRESSION FOR THE PERCENTAGE POINTS OF V_1^*

		q=4	p=1	q=5	p=1
n	α	L-C-K	D-F	L-C-K	D-F
10	0.05	3.238	3.238	4.978	4.977
10	0.01	4.330	4.331	6.333	6.333
15	0.05	1.967	1.967	2.949	2.948
15	0.01	2.628	2.628	3.742	3.742
20	0.05	1.414	1.414	2.099	2.099
20	0.01	1.888	1.888	2.663	2.663
24	0.05	1.154	1.154	1.706	1.707
24	0.01	1.541	1.542	2.165	2.164

TABLE III*

PERCENTAGE POINTS OF $-2 \log V_1$

q=3

	p=1			p=2			p=3		
M \ α	10%	5%	1%	10%	5%	1%	10%	5%	1%
1	2.419	3.023	4.398	5.686	6.371	7.991	9.161	10.037	11.838
2	2.027	2.534	3.678	4.851	5.589	6.899	8.091	8.857	10.427
3	1.744	2.180	3.165	4.279	4.857	6.076	7.255	7.937	9.334
4	1.531	1.913	2.778	3.829	4.345	5.432	6.582	7.198	8.457
5	1.363	1.704	2.475	3.466	3.933	4.914	6.026	6.587	7.735
6	1.229	1.537	2.231	3.167	3.593	4.487	5.559	6.075	7.130
7	1.119	1.400	2.031	2.916	3.307	4.129	5.160	5.639	6.615
8	1.027	1.284	1.864	2.701	3.064	3.825	4.815	5.262	6.171
9	.949	1.187	1.723	2.517	2.854	3.563	4.515	4.933	5.784
10	.882	1.103	1.602	2.356	2.672	3.335	4.250	4.643	5.443
11	.824	1.030	1.496	2.215	2.511	3.134	4.016	4.386	5.142
12	.773	.967	1.403	2.089	2.369	2.956	3.805	4.157	4.871
13	.728	.910	1.322	1.977	2.242	2.798	3.616	3.950	4.629
14	.688	.850	1.249	1.877	2.128	2.655	3.445	3.763	4.409
15	.652	.815	1.183	1.786	2.025	2.527	3.289	3.593	4.210
16	.620	.775	1.125	1.704	1.932	2.410	3.148	3.438	4.028
17	.591	.739	1.072	1.629	1.847	2.304	3.018	3.296	3.861
18	.564	.705	1.024	1.561	1.769	2.207	2.898	3.165	3.707
19	.540	.675	.979	1.497	1.697	2.117	2.788	3.044	3.566
20	.517	.647	.939	1.439	1.631	2.035	2.685	2.932	3.434
22	.478	.597	.867	1.335	1.513	1.888	2.501	2.732	3.199
24	.444	.555	.805	1.245	1.412	1.761	2.341	2.557	2.995
26	.414	.518	.752	1.167	1.323	1.650	2.200	2.403	2.814
28	.388	.486	.705	1.098	1.244	1.552	2.076	2.267	2.655
30	.366	.458	.664	1.036	1.175	1.465	1.965	2.145	2.512

* The entries in this table are the values of c_1 where $M = n - s - 3$ and $p[-2 \log V_1 \leq c_1] = (1 - \alpha)$.

TABLE III (Continued)

q = 4

	p=1			p=2			p=3		
M \ α	10%	5%	1%	10%	5%	1%	10%	5%	1%
1	3.694	4.376	5.861	8.557	9.417	11.195	13.949	14.932	16.905
2	3.143	3.721	4.980	7.534	8.285	9.831	12.539	13.408	15.155
3	2.736	3.238	4.330	6.739	7.406	8.777	11.407	12.190	13.762
4	2.422	2.867	3.832	6.100	6.701	7.936	10.473	11.189	12.618
5	2.174	2.572	3.438	5.574	6.122	7.245	9.689	10.347	11.660
6	1.971	2.333	3.118	5.134	5.637	6.669	9.017	9.627	10.846
7	1.804	2.134	2.852	4.769	5.225	6.179	8.436	9.004	10.140
8	1.663	1.967	2.628	4.436	4.870	5.757	7.927	8.460	9.524
9	1.542	1.825	2.437	4.154	4.560	5.391	7.477	7.980	8.981
10	1.438	1.701	2.272	3.907	4.289	5.068	7.078	7.552	8.498
11	1.347	1.593	2.128	3.688	4.048	4.783	6.719	7.169	8.066
12	1.266	1.498	2.001	3.493	3.833	4.528	6.395	6.823	7.676
13	1.195	1.414	1.888	3.317	3.639	4.299	6.103	6.510	7.323
14	1.132	1.338	1.788	3.158	3.465	4.093	5.835	6.225	7.001
15	1.074	1.271	1.697	3.013	3.306	3.905	5.591	5.964	6.708
16	1.023	1.210	1.616	2.882	3.162	3.734	5.367	5.725	6.437
17	.976	1.154	1.541	2.761	3.030	3.578	5.160	5.503	6.188
18	.933	1.104	1.474	2.650	2.908	3.434	4.968	5.299	5.958
19	.894	1.057	1.412	2.548	2.796	3.302	4.790	5.110	5.745
20	.858	1.015	1.355	2.453	2.692	3.179	4.626	4.933	5.546
22	.794	.939	1.254	2.284	2.506	2.959	4.327	4.615	5.188
24	.739	.874	1.167	2.137	2.344	2.768	4.066	4.336	4.874
26	.691	.817	1.091	2.007	2.202	2.600	3.833	4.088	4.596
28	.649	.767	1.025	1.893	2.076	2.451	3.627	3.868	4.348
30	.612	.723	.966	1.790	1.964	2.318	3.442	3.671	4.125

TABLE III (Continued)

q = 5

M \ α	p=1 10%	p=1 5%	p=1 1%	p=2 10%	p=2 5%	p=2 1%	p=3 10%	p=3 5%	p=3 1%
1	5.039	5.784	7.368	11.665	12.600	14.500	18.945	20.006	22.106
2	4.339	4.978	6.333	10.405	11.228	12.899	17.242	18.192	20.067
3	3.813	4.371	5.557	9.405	10.146	11.639	15.852	16.716	18.421
4	3.402	3.900	4.954	8.589	9.261	10.616	14.687	15.477	17.043
5	3.071	3.520	4.470	7.909	8.525	9.767	13.691	14.425	15.876
6	2.801	3.208	4.073	7.331	7.900	9.047	12.828	13.515	14.864
7	2.573	2.949	3.742	6.835	7.364	8.430	12.073	12.717	13.981
8	2.381	2.727	3.461	6.402	6.897	7.893	11.406	12.011	13.204
9	2.215	2.537	3.219	6.023	6.487	7.423	10.811	11.386	12.513
10	2.071	2.372	3.009	5.686	6.125	7.006	10.279	10.821	11.890
11	1.944	2.227	2.826	5.386	5.801	6.635	9.798	10.313	11.332
12	1.832	2.099	2.663	5.116	5.510	6.301	9.360	9.852	10.822
13	1.733	1.985	2.518	4.873	5.247	6.000	8.960	9.432	10.359
14	1.644	1.882	2.388	4.652	5.009	5.727	8.594	9.047	9.935
15	1.563	1.790	2.270	4.449	4.791	5.477	8.259	8.691	9.545
16	1.490	1.706	2.165	4.265	4.592	5.249	7.948	8.364	9.185
17	1.424	1.631	2.068	4.095	4.409	5.039	7.661	8.061	8.850
18	1.363	1.561	1.980	3.938	4.240	4.846	7.393	7.779	8.541
19	1.307	1.497	1.898	3.793	4.084	4.667	7.144	7.518	8.252
20	1.256	1.438	1.823	3.658	3.938	4.501	6.911	7.272	7.983
22	1.164	1.333	1.690	3.415	3.677	4.202	6.498	6.828	7.495
24	1.085	1.243	1.576	3.203	3.448	3.940	6.116	6.435	7.063
26	1.016	1.163	1.475	3.016	3.247	3.709	5.784	6.086	6.679
28	.955	1.094	1.387	2.849	3.067	3.504	5.486	5.772	6.335
30	.901	1.032	1.309	2.700	2.906	3.321	5.219	5.491	6.025

SECTION IV

LIKELIHOOD RATIO TEST STATISTIC FOR SPHERICITY

The likelihood ratio statistic for testing $H_2 : \Sigma = \sigma^2 \Sigma_o$, where σ^2 is unknown and Σ_o is known, is given by

$$V_2 = \frac{\left|A \Sigma_o^{-1}\right|}{\left\{\mathrm{tr}\, A \Sigma_o^{-1}/s\right\}^s} \tag{9}$$

where A was defined in Section III and tr B denotes the trace of B. The h-th moment of V_2 is given by

$$E(V_2^h) = \frac{p^{hs}\Gamma(sn/2)}{\Gamma(sh + \frac{sn}{2})} \prod_{i=1}^{s} \frac{\Gamma\left[\frac{1}{2}(n+1-i)+h\right]}{\Gamma\left[\frac{1}{2}(n+1-i)\right]} \tag{10}$$

The statistic V_2 and its moments were derived by Mauchly [11]. Using the first four moments, we approximated the distribution of $V_2^{1/b}$ with the Pearson Type I distribution, where b is a suitably chosen integer. Expressions for the exact distribution of V_2 were given by Consul [6], Mathai and Rathie [12], and Nagarsenker and Pillai [13], but these expressions are very difficult to compute. In Table IV the values under the column LCK are the values obtained by the authors using the Pearson type approximation whereas the corresponding exact values are taken from Nagarsenker and Pillai [13]. This table indicates that the accuracy of the Pearson type approximation is sufficient for practical purposes. The values of α in this table are given by the relation

$$P\left[V_2 \geq c_2 | H_2\right] = (1 - \alpha)$$

TABLE IV

COMPARISON OF THE PEARSON TYPE APPROXIMATION WITH EXACT EXPRESSION FOR THE DISTRIBUTION OF V_2

		s=4		s=5		s=7	
n	α	L-C-K	Exact	L-C-K	Exact	L-C-K	Exact
6	0.05	0.0169	0.0169	0.0013	0.0013	--	--
6	0.01	0.0050	0.0050	0.00022	0.00022	--	--
10	0.05	0.1297	0.1297	0.0492	0.0492	0.0029	0.0030
10	0.01	0.0726	0.0726	0.0242	0.0242	0.0010	0.0010
15	0.05	0.2812	0.2812	0.1608	0.1608	0.0368	0.0368
15	0.01	0.1967	0.1966	0.1052	0.1052	0.0207	0.0207
21	0.05	0.4173	0.4173	0.2877	0.2876	0.1111	0.1111
21	0.01	0.3264	0.3264	0.2156	0.2156	0.0761	0.0761
33	0.05	0.5833	0.5833	0.4663	0.4663	0.2665	0.2665
33	0.01	0.5013	0.5013	0.3910	0.3910	0.2125	0.2125
41	0.05	0.6507	0.6508	0.5453	0.5453	0.3515	0.3515
41	0.01	0.5769	0.5769	0.4741	0.4741	0.2939	0.2939

SECTION V

LIKELIHOOD RATIO STATISTIC FOR TESTING THE HYPOTHESIS THAT THE COVARIANCE MATRIX IS EQUAL TO A SPECIFIED MATRIX

Consider the problem of testing the hypothesis $H_3 : \Sigma = \Sigma_o$ where Σ_o is specified. The likelihood ratio statistic for testing H_3 and the moments of this statistic were derived by Anderson [14]. The modified likelihood ratio test statistic (obtained by changing N to n in the likelihood ratio statistic) and its moments are as given below:

$$V_3 = (e/n)^{sn/2} \left| A\, \Sigma_o^{-1} \right|^{n/2} \operatorname{etr}\left(-\frac{1}{2} A\, \Sigma_o^{-1}\right) \tag{11}$$

$$E(V_3^h) = (2e/n)^{shn/2} \left| \Sigma_o \right|^{nh/2}$$

$$\cdot \left| I + h\, \Sigma_o \right|^{-n(1+h)/2} \frac{\prod_{i=1}^{s} \Gamma\left[\frac{1}{2}(n+nh+1-i)\right]}{\prod_{i=1}^{s} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \tag{12}$$

Using the first four moments, we approximated the distribution of $V_3^{1/b}$ with the Pearson Type I distribution as in Section III, where b is a suitably chosen integer. Korin [15] obtained an asymptotic expression of order n^{-15} for the distribution of $V_3^* = -2 \log V_3$. Using this expression, he computed percentage points of V_3^* for some values of the parameters. Nagarsenker and Pillai [16] obtained an expression for the distribution of V_3^*, but this expression is complicated from a computational point of view. Using this expression, they computed exact percentage points of V_3^*. In Table V we compare our values (given in the column L-C-K) obtained by using the Pearson type approximation with exact

values of Nagarsenker and Pillai, and the values obtained by Korin [15]. Table V indicates that the accuracy of the Pearson type approximation is sufficient for practical purposes.

TABLE V

COMPARISON OF THE PEARSON TYPE APPROXIMATION

WITH EXACT AND ASYMPTOTIC EXPRESSIONS

FOR THE DISTRIBUTION OF V_3^*

s=4					s=6					s=10			
n	α	L-C-K	Exact	Korin	n	α	L-C-K	Exact	Korin	n	α	L-C-K	Exact
6	0.05	25.76	25.76	25.8	8	0.05	49.24	49.25	--	12	0.05	119.08	119.07
6	0.01	33.07	33.08	--	8	0.01	59.59	59.60	--	12	0.01	135.81	135.81
7	0.05	24.06	24.06	24.06	9	0.05	45.82	45.83	--	13	0.05	111.15	111.15
7	0.01	30.74	30.75	30.8	9	0.01	55.15	55.16	--	13	0.01	126.10	126.10
10	0.05	21.75	21.75	21.75	10	0.05	43.62	43.63	--	14	0.05	105.76	105.76
10	0.01	27.66	27.66	27.66	10	0.01	52.35	52.35	--	14	0.01	119.64	119.64
11	0.05	21.35	21.35	21.35	15	0.05	38.71	38.71	--	15	0.05	101.83	101.82
11	0.01	27.13	27.13	27.13	15	0.01	46.23	46.23	--	15	0.01	114.96	114.97
13	0.05	20.77	20.77	20.77	20	0.05	36.87	36.86	36.87	20	0.05	91.28	91.28
13	0.01	26.38	--	--	20	0.01	43.97	43.98	43.99	20	0.01	102.71	102.70
					25	0.05	35.89	35.88	35.89	25	0.05	86.51	86.52
					25	0.01	42.79	42.79	42.80	25	0.01	97.23	97.23

SECTION VI

LIKELIHOOD RATIO STATISTIC FOR TESTING THE HOMOGENEITY OF COVARIANCE MATRICES

Let $\underset{\sim}{X}' = (\underset{\sim}{X}_1',\ldots,\underset{\sim}{X}_q')$ be as defined in Section II. Let $\underset{\sim}{X}_{ij}$ $(j=1,\ldots,N_i)$ be j-th independent observation on $\underset{\sim}{X}_i$. Also, let $p_i = p(i = 1,\ldots,q)$ and

$$A_{ii} = \sum_{j=1}^{N_i} (\underset{\sim}{X}_{ij} - \overline{\underset{\sim}{X}}_i)(\underset{\sim}{X}_{ij} - \overline{\underset{\sim}{X}}_i)'.$$

Wilks [1] derived the likelihood ratio statistic for $H_4 : \Sigma_{11} = \ldots = \Sigma_{qq}$ when $\Sigma_{ij} = 0(i \neq j=1,..q)$ and derived its moments. Let $n_i = N_i-1$, and $n = \sum_{i=1}^{q} n_i$. The modified likelihood ratio statistic V_4 (obtained by interchanging N_i with n_i in the likelihood ratio statistic) for H_4 and the moments of V_4 are given below:

$$V_4 = \frac{\prod_{g=1}^{q} \left| A_{gg} \right|^{n_g/2}}{\left| \sum_{g=1}^{q} A_{gg} \right|^{n/2}} \quad \frac{n^{pn/2}}{\prod_{g=1}^{q} n_g^{pn_g/2}} \tag{13}$$

$$E(V_4^h) = \left(n^{phn/2} \Big/ \prod_{g=1}^{q} n_g^{phn_g/2} \right)$$

$$\cdot \prod_{i=1}^{p} \left[\left\{ \prod_{g=1}^{q} \Gamma[(n_g+hn_g+1-i)/2]/\Gamma[(n_g+1-i)/2] \right\} \right.$$

$$\left. \cdot \left\{ \Gamma[((n+1-i)/2]/\Gamma[(n+hn+1-i)/2] \right\} \right] \tag{14}$$

Korin [17] computed percentage points by using Box's asymptotic expression up to terms of order n^{-15}. Davis and Field [8] computed the percentage points by using the Cornish-Fisher type inversion of Box's asymptotic expression. In this paper, we approximated the distribution of $V_4^{1/b}$, with a suitable Pearson type distribution where b is a suitably chosen integer. The type of distribution that is fitted is the Pearson's Type I distribution. In Table VI, we compared some of the percentage points obtained by us with the corresponding values obtained by Davis and Field [8] and Korin [17], where $n_i = n_o$. The percentage points c_4 obtained by us using the Pearson type approximation are given under the column L-C-K whereas the corresponding values obtained by Korin, and Davis-Field are given under the columns Korin and D-F, respectively. Table VI indicates that the Pearson type approximation is satisfactory for practical purposes. In this table, α is defined by the equation

$$P\left[V_4^* \leq c_4 | H_4\right] = (1 - \alpha) \tag{15}$$

When n_o is large, Bishop [18] investigated the accuracy of approximating the distribution of the $\frac{2}{N}$th power of the likelihood ratio criterion with the Beta distribution for a few cases, where $n = qn_o$.

Using the Pearson type approximation, we computed the values of c_4 when $n_o = (p+1)(1)20(5)30$, $p = 2(1)6$, $q = 2(1)10$ and $\alpha = 0.01, 0.05, 0.10$. These percentage points are given in Table VII.

TABLE VI

COMPARISON OF THE PEARSON TYPE APPROXIMATION

WITH THE ASYMPTOTIC EXPRESSION FOR

THE DISTRIBUTION OF V_4^*

$\alpha = 0.05$

		p=2			p=3			p=6		
n_o	k	L-C-K	Korin	D-F	L-C-K	Korin	D-F	L-C-K	Korin	D-F
4	2	10.70	10.70	10.70	22.41	--	--	--	--	--
4	9	46.07	46.07	--	99.94	--	--	--	--	--
7	2	9.24	9.24	9.24	16.59	16.59	16.59	69.62	--	--
7	9	41.26	41.26	--	79.90	79.91	--	345.81	--	--
10	2	8.76	8.76	8.76	15.11	15.11	15.11	49.95	49.95	49.95
10	9	39.65	36.65*	--	74.58	74.57	--	271.84	--	--
15	2	8.42	--	8.42	14.15	--	14.15	42.03	42.03	42.02
15	9	38.50	--	--	71.05	--	--	239.72	--	--
20	2	8.26	--	8.26	13.72	--	13.72	39.11	39.11	39.10
20	9	37.95	--	--	69.45	--	--	227.45	--	--

* There is a typographical error in Korin's table. The correct value seems to be 39.65.

TABLE VII

PERCENTAGE POINTS OF THE DISTRIBUTION

OF -2 log V_4

$\alpha = 0.10 \qquad p = 2$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
3	9.75	15.80	21.29	26.52	31.50	36.58	41.49	46.32	51.12
4	8.56	14.07	19.08	23.87	28.52	33.07	37.55	41.97	46.36
5	7.97	13.22	17.99	22.55	26.37	31.31	35.58	39.80	43.97
6	7.63	12.70	17.33	21.75	26.05	30.25	34.39	38.49	42.53
7	7.39	12.36	16.89	21.22	25.42	29.54	33.60	37.61	41.57
8	7.23	12.11	16.57	20.84	24.98	29.04	33.03	36.98	40.88
9	7.10	11.93	16.34	20.55	24.65	28.66	32.61	36.51	40.37
10	7.01	11.78	16.16	20.33	24.39	28.36	32.28	36.14	39.97
11	6.93	11.67	16.01	20.15	24.18	28.12	32.01	35.85	39.65
12	6.87	11.58	15.88	20.01	24.01	27.93	31.79	35.61	39.38
13	6.82	11.50	15.79	19.89	23.87	27.77	31.62	35.41	39.16
14	6.77	11.43	15.70	19.78	23.75	27.63	31.46	35.24	38.98
15	6.74	11.37	15.63	19.69	23.65	27.52	31.33	35.09	38.82
16	6.71	11.33	15.57	19.62	23.56	27.42	31.21	34.97	38.68
17	6.68	11.28	15.51	19.55	23.48	27.32	31.11	34.86	38.56
18	6.65	11.25	15.46	19.49	23.41	27.25	31.03	34.70	38.45
19	6.63	11.21	15.42	19.44	23.35	27.18	30.95	34.68	38.36
20	6.61	11.18	15.38	19.40	23.30	27.12	30.88	34.60	38.28
25	6.53	11.07	15.24	19.22	23.09	26.88	30.61	34.30	37.95
30	6.48	11.00	15.14	19.10	22.96	26.73	30.45	34.12	37.74

$\alpha = 0.10 \qquad p = 3$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
4	18.87	30.76	41.78	52.41	62.90	73.03	83.14	93.16	103.12
5	16.19	26.87	36.79	46.36	55.73	64.95	74.05	83.09	92.05
6	14.83	24.87	34.22	43.24	52.06	60.76	69.35	77.86	86.30
7	14.01	23.66	32.64	41.33	49.91	58.18	66.44	74.64	82.76
8	13.46	22.83	31.58	40.03	48.29	56.44	64.48	72.45	80.37
9	13.06	22.24	30.81	39.09	47.19	55.17	63.05	70.88	78.64
10	12.77	21.79	30.23	38.38	46.36	54.21	61.95	69.68	77.31
11	12.53	21.44	29.77	37.82	45.70	53.46	61.13	68.74	76.26
12	12.35	21.16	29.40	37.37	45.17	52.86	60.46	67.98	75.45
13	12.19	20.93	29.10	37.00	44.74	52.36	59.83	67.35	74.77
14	12.07	20.73	28.85	36.69	44.36	51.94	59.42	66.84	74.19
15	11.96	20.57	28.63	36.43	44.06	51.59	59.03	66.39	73.70
16	11.87	20.43	28.45	36.21	43.80	51.28	58.67	66.01	73.29
17	11.78	20.30	28.29	36.01	43.57	51.02	58.38	65.67	72.92
18	11.71	20.20	28.15	35.83	43.37	50.79	58.11	65.38	72.59
19	11.65	20.10	28.02	35.68	43.19	50.58	57.88	65.12	72.32
20	11.60	20.02	27.91	35.55	43.02	50.39	57.68	64.90	72.07
25	11.39	19.70	27.50	35.05	42.43	49.71	56.91	64.04	71.12
30	11.26	19.50	27.23	34.72	42.05	49.27	56.42	63.49	70.52

TABLE VII (Continued)

$\alpha = 0.10 \quad p = 4$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
5	30.70	50.45	68.96	86.91	104.53	121.93	139.15	156.25	173.24
6	26.16	43.85	60.44	76.56	92.38	107.98	123.46	138.81	154.09
7	23.79	40.34	55.91	71.02	85.37	100.51	115.03	129.45	143.75
8	22.33	38.15	53.06	67.55	81.76	95.81	109.73	123.52	137.25
9	21.33	36.65	51.11	65.15	78.34	92.55	106.06	119.46	132.76
10	20.60	35.56	49.67	63.39	76.37	90.17	103.37	116.45	129.48
11	20.06	34.72	48.58	62.04	75.28	88.36	101.30	114.15	126.93
12	19.62	34.06	47.71	60.98	74.02	86.90	99.68	112.34	124.95
13	19.27	33.53	47.00	60.12	73.00	85.73	98.36	110.88	123.31
14	18.98	33.08	46.42	59.40	72.17	84.76	97.26	109.65	121.98
15	18.74	32.71	45.94	58.80	71.45	83.95	96.32	108.61	120.85
16	18.53	32.40	45.52	58.30	70.34	83.25	95.55	107.73	119.87
17	18.36	32.12	45.16	57.65	70.33	82.65	94.86	106.98	119.03
18	18.21	31.89	44.85	57.47	69.87	82.13	94.26	106.32	118.30
19	18.07	31.68	44.58	57.13	69.47	81.66	93.75	105.73	117.67
20	17.96	31.49	44.33	56.83	69.12	81.25	93.28	105.22	117.10
25	17.51	30.81	43.43	55.72	67.30	79.75	91.57	103.33	115.00
30	17.24	30.37	42.86	55.02	66.37	78.79	90.49	102.11	113.67

$\alpha = 0.10 \quad p = 5$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
6	45.25	74.90	102.86	130.09	156.89	183.39	209.65	235.79	261.72
7	38.56	65.12	90.21	114.66	138.70	162.50	186.15	209.58	232.90
8	34.95	59.77	83.27	106.16	128.70	150.99	173.12	195.06	216.94
9	32.69	56.37	78.81	100.70	122.23	143.54	164.69	185.70	206.61
10	31.12	54.00	75.70	96.88	117.72	138.36	158.82	179.16	199.37
11	29.97	52.24	73.40	94.03	114.37	134.47	154.42	174.28	194.00
12	29.08	50.88	71.62	91.85	111.78	131.47	151.03	170.52	189.94
13	28.39	49.81	70.20	90.09	109.70	129.10	148.33	167.49	186.53
14	27.82	48.93	69.04	88.66	108.01	127.14	146.14	165.04	183.78
15	27.35	48.21	68.07	87.48	106.60	125.51	144.31	162.94	181.54
16	26.95	47.59	67.26	86.47	105.42	124.14	142.75	161.24	179.62
17	26.62	47.07	66.57	85.62	104.40	122.98	141.42	159.75	177.97
18	26.32	46.61	65.96	84.87	103.50	121.95	140.27	158.45	176.56
19	26.07	46.21	65.44	84.22	102.74	121.06	139.23	157.32	175.30
20	25.85	45.35	64.97	83.65	102.05	120.28	138.36	156.32	174.20
25	25.03	44.59	63.28	81.56	99.57	117.41	135.10	152.67	170.19
30	24.52	43.78	62.22	80.23	98.01	115.60	133.04	150.39	167.69

TABLE VII (Continued)

$\alpha = 0.10 \quad p = 6$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
7	62.54	104.12	143.53	181.97	219.37	257.40	294.64	331.76	368.58
8	53.42	90.76	126.20	160.82	194.34	228.72	262.28	295.59	328.81
9	48.40	83.29	116.44	148.84	180.79	212.41	243.93	275.04	306.13
10	45.18	78.42	110.07	141.01	171.49	201.73	231.72	261.56	291.26
11	42.92	74.99	105.56	135.44	164.91	194.13	223.14	251.93	280.69
12	41.23	72.42	102.16	131.26	159.38	188.42	216.65	244.70	272.69
13	39.94	70.43	99.54	128.00	156.11	183.93	211.57	239.10	266.48
14	38.91	68.83	97.42	125.39	153.01	180.35	207.54	234.56	261.46
15	38.06	67.51	95.67	123.24	150.46	177.39	204.16	230.78	257.33
16	37.36	66.42	94.23	121.46	148.31	174.95	201.38	227.67	253.85
17	36.76	65.49	93.00	119.93	146.51	172.84	199.02	225.05	250.97
18	36.26	64.59	91.93	118.60	144.95	171.03	196.97	222.74	248.44
19	35.82	64.00	91.01	117.47	143.59	169.49	195.17	220.73	246.20
20	35.43	63.39	90.21	116.47	142.40	168.11	193.61	219.00	244.29
25	34.06	61.22	87.30	112.86	138.11	163.13	187.98	212.70	237.33
30	33.21	59.97	85.49	110.63	135.45	160.06	184.51	208.78	233.04

$\alpha = 0.05 \quad p = 2$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
3	12.18	18.70	24.55	30.09	35.45	40.68	45.81	50.87	55.86
4	10.70	16.65	22.00	27.07	31.97	36.75	41.45	46.07	50.64
5	9.97	15.63	20.73	25.57	30.23	34.79	39.26	43.67	48.02
6	9.53	15.02	19.97	24.66	29.19	33.61	37.95	42.22	46.45
7	9.24	14.62	19.46	24.05	28.49	32.83	37.08	41.26	45.40
8	9.04	14.33	19.10	23.62	27.99	32.26	36.44	40.57	44.64
9	8.88	14.11	18.83	23.30	27.62	31.84	35.98	40.05	44.08
10	8.76	13.94	18.61	23.05	27.33	31.51	35.61	39.65	43.64
11	8.67	13.81	18.44	22.85	27.10	31.25	35.32	39.33	43.29
12	8.59	13.70	18.30	22.68	26.90	31.03	35.08	39.07	43.00
13	8.52	13.60	18.19	22.54	26.75	30.85	34.87	38.84	42.76
14	8.47	13.53	18.10	22.42	26.61	30.70	34.71	38.66	42.56
15	8.42	13.46	18.01	22.33	26.50	30.57	34.57	38.50	42.38
16	8.38	13.40	17.94	22.24	26.40	30.45	34.43	38.36	42.23
17	8.35	13.35	17.87	22.17	26.31	30.35	34.32	38.24	42.10
18	8.32	13.30	17.82	22.10	26.23	30.27	34.23	38.13	41.99
19	8.28	13.26	17.77	22.04	26.16	30.19	34.14	38.04	41.88
20	8.26	13.23	17.72	21.98	26.10	30.12	34.07	37.95	41.79
25	8.17	13.10	17.55	21.79	25.87	29.86	33.78	37.63	41.44
30	8.11	13.01	17.44	21.65	25.72	29.69	33.59	37.42	41.21

TABLE VII (Continued)

$\alpha = 0.05 \quad p = 3$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
4	22.41	35.00	46.58	57.68	68.50	79.11	89.60	99.94	110.21
5	19.19	30.52	40.95	50.95	60.59	70.26	79.69	89.03	98.27
6	17.57	28.24	38.06	47.49	56.57	65.69	74.58	83.39	92.09
7	16.59	26.84	36.29	45.37	54.20	62.89	71.44	79.90	89.30
8	15.93	25.90	35.10	43.93	52.54	60.99	69.32	77.57	85.73
9	15.46	25.22	34.24	42.90	51.33	59.62	67.78	75.86	83.87
10	15.11	24.71	33.59	42.11	50.42	58.57	66.62	74.58	82.46
11	14.83	24.31	33.08	41.50	49.71	57.76	65.71	73.57	81.36
12	14.61	23.99	32.67	41.00	49.13	57.11	64.97	72.75	80.45
13	14.43	23.73	32.33	40.60	48.65	56.56	64.36	72.09	79.72
14	14.28	23.50	32.05	40.26	48.26	56.11	63.86	71.53	79.11
15	14.15	23.32	31.81	39.97	47.92	55.73	63.43	71.05	78.60
16	14.04	23.16	31.60	39.72	47.63	55.40	63.06	70.64	78.14
17	13.94	23.02	31.43	39.50	47.38	55.11	62.73	70.27	77.76
18	13.86	22.89	31.26	39.31	47.16	54.86	62.45	69.97	77.41
19	13.79	22.78	31.13	39.15	46.96	54.64	62.21	69.69	77.11
20	13.72	22.69	31.01	39.00	46.79	54.44	61.98	69.45	76.84
25	13.48	22.33	30.55	38.44	46.15	53.70	61.16	68.54	75.84
30	13.32	22.10	30.25	38.09	45.73	53.22	60.62	67.94	75.18

$\alpha = 0.05 \quad p = 4$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
5	35.39	56.10	75.36	93.97	112.17	130.11	147.81	165.39	182.80
6	30.06	48.62	65.90	82.60	98.93	115.03	130.94	146.69	162.34
7	27.31	44.69	60.89	76.56	91.88	106.98	121.90	136.71	151.39
8	25.61	42.24	57.77	72.77	87.46	101.94	116.23	130.43	144.50
9	24.45	40.57	55.62	70.17	84.42	98.46	112.32	126.08	139.74
10	23.62	39.34	54.04	68.26	82.19	95.90	109.46	122.91	136.24
11	22.98	38.41	52.84	66.81	80.48	93.95	107.27	120.46	133.57
12	22.48	37.67	51.90	65.66	79.14	92.41	105.54	118.55	131.45
13	22.08	37.08	51.13	64.73	78.04	91.15	104.12	116.98	129.74
14	21.75	36.59	50.50	63.95	77.13	90.12	102.97	115.69	128.32
15	21.47	36.17	49.97	63.30	76.37	89.26	101.93	114.59	127.14
16	21.24	35.82	49.51	62.76	75.73	88.51	101.14	113.67	126.10
17	21.03	35.52	49.12	62.28	75.16	87.87	100.42	112.87	125.22
18	20.86	35.26	48.78	61.86	74.68	87.31	99.80	112.17	124.46
19	20.70	35.02	48.47	61.50	74.25	86.82	99.25	111.56	123.79
20	20.56	34.82	48.21	61.17	73.87	86.38	98.75	111.02	123.18
25	20.06	34.06	47.23	59.98	72.47	84.78	96.95	109.01	120.99
30	19.74	33.59	46.61	59.21	71.58	83.74	95.79	107.71	119.57

TABLE VII (Continued)

$\alpha = 0.05 \quad p = 5$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
6	51.11	81.99	110.92	138.98	166.54	193.71	220.66	247.37	273.88
7	43.40	71.06	97.03	122.22	146.95	171.34	195.43	219.47	243.30
8	39.29	65.15	89.45	113.03	136.18	159.04	181.65	204.14	226.48
9	36.71	61.39	84.62	107.17	129.30	151.17	172.80	194.27	215.64
10	34.93	58.78	81.25	103.06	124.48	145.64	166.56	187.37	208.02
11	33.62	56.85	78.75	100.02	120.92	141.54	161.99	182.24	202.37
12	32.62	55.37	76.83	97.68	118.15	138.38	158.38	178.23	198.03
13	31.83	54.19	75.30	95.82	115.36	135.86	155.54	175.10	194.51
14	31.19	53.23	74.05	94.29	114.16	133.80	153.21	172.49	191.68
15	30.66	52.44	73.01	93.02	112.56	132.07	151.29	170.36	189.30
16	30.22	51.76	72.14	91.94	111.41	130.61	149.56	168.53	187.32
17	29.83	51.19	71.39	91.03	110.34	129.38	148.25	166.99	185.61
18	29.51	50.69	70.74	90.23	109.39	128.29	147.03	165.65	184.10
19	29.22	50.26	70.17	89.54	108.57	127.36	145.97	164.45	182.81
20	28.97	49.88	69.67	88.93	107.35	126.52	145.02	163.38	181.65
25	28.05	48.48	67.86	86.70	105.21	123.51	141.62	159.60	177.49
30	27.48	47.61	66.71	85.29	103.56	121.60	139.47	157.22	174.67

$\alpha = 0.05 \quad p = 6$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
7	69.62	112.69	153.28	192.75	231.50	269.98	308.03	345.81	383.45
8	59.24	97.92	134.42	169.94	204.36	239.37	273.52	307.61	341.44
9	53.56	89.72	123.87	157.11	189.33	222.14	254.18	286.00	317.69
10	49.95	84.42	117.03	148.77	179.37	210.85	241.48	271.84	302.13
11	47.43	80.69	112.18	142.85	173.10	202.85	232.46	261.80	291.03
12	45.56	77.90	108.56	138.40	167.79	196.84	225.65	254.25	282.76
13	44.11	75.74	105.72	134.94	163.72	192.15	220.37	248.39	276.24
14	42.96	74.00	103.46	132.19	160.44	188.39	216.09	243.67	271.04
15	42.03	72.59	101.61	129.90	157.76	185.31	212.63	239.72	266.75
16	41.24	71.41	100.07	128.01	155.52	182.73	209.70	236.48	263.15
17	40.58	70.40	98.76	126.38	153.60	180.51	207.23	233.73	260.07
18	40.02	69.54	97.63	125.01	151.36	178.63	205.09	231.36	257.48
19	39.53	68.80	96.64	123.80	150.55	176.99	203.21	229.26	255.19
20	39.11	68.13	95.77	122.73	149.29	175.53	201.60	227.45	253.19
25	37.58	65.79	92.69	118.93	144.78	170.35	195.69	220.92	246.00
30	36.63	64.33	90.77	116.57	141.37	167.13	192.04	216.82	241.49

TABLE VII (Continued)

$\alpha = 0.01 \quad p = 2$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
3	17.67	24.99	31.49	37.60	43.45	49.14	54.70	60.16	65.52
4	15.53	22.25	28.20	33.80	39.15	44.36	49.44	54.43	59.35
5	14.46	20.38	26.56	31.90	37.01	41.97	46.82	51.57	56.26
6	13.84	20.07	25.58	30.76	35.72	40.54	45.24	49.86	54.40
7	13.42	19.52	24.93	30.00	34.36	39.59	44.20	48.71	53.16
8	13.12	19.13	24.46	29.46	34.25	38.90	43.44	47.89	52.28
9	12.90	18.84	24.11	29.05	33.80	38.39	42.85	47.28	51.61
10	12.72	18.61	23.84	28.74	33.44	37.99	42.44	46.80	51.10
11	12.58	18.43	23.62	28.49	33.15	37.68	42.09	46.42	50.69
12	12.47	18.29	23.44	28.28	32.92	37.41	41.80	46.11	50.35
13	12.37	18.16	23.30	28.11	32.72	37.20	41.57	45.85	50.07
14	12.30	18.06	23.17	27.97	32.56	37.01	41.36	45.63	49.83
15	12.23	17.97	23.07	27.84	32.42	36.85	41.19	45.44	49.62
16	12.17	17.89	22.97	27.73	32.29	36.72	41.04	45.28	49.45
17	12.12	17.83	22.89	27.64	32.18	36.60	40.91	45.13	49.29
18	12.07	17.77	22.82	27.56	32.09	36.49	40.79	45.01	49.16
19	12.03	17.71	22.75	27.48	32.01	36.40	40.69	44.89	49.03
20	11.99	17.67	22.69	27.42	31.93	36.32	40.60	44.79	48.93
25	11.86	17.49	22.48	27.16	31.65	36.00	40.25	44.42	48.52
30	11.77	17.37	22.34	27.00	31.47	35.80	40.02	44.17	48.25

$\alpha = 0.01 \quad p = 3$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
4	30.17	44.00	56.58	68.55	80.14	91.49	102.60	113.57	124.41
5	25.75	38.23	49.56	60.36	70.30	81.01	91.03	100.91	110.68
6	23.53	35.30	46.00	56.17	66.02	75.65	85.10	94.42	103.61
7	22.20	33.53	43.83	53.62	63.11	72.38	81.47	90.45	99.30
8	21.31	32.34	42.37	51.91	61.14	70.17	79.03	87.76	96.38
9	20.67	31.48	41.32	50.67	59.73	68.58	77.25	85.82	94.27
10	20.19	30.84	40.52	49.73	58.65	67.37	75.91	84.34	92.69
11	19.82	30.33	39.90	49.01	57.81	66.42	74.86	83.20	91.43
12	19.52	29.93	39.40	48.42	57.14	65.66	74.03	82.28	90.42
13	19.28	29.60	39.00	47.93	56.58	65.03	73.33	81.51	89.59
14	19.07	29.32	38.65	47.53	56.13	64.51	72.75	80.87	88.90
15	18.90	29.06	38.36	47.18	55.73	64.07	72.27	80.34	88.31
16	18.75	28.88	38.11	46.89	55.39	63.69	71.83	79.86	87.79
17	18.62	28.71	37.89	46.63	55.10	63.36	71.47	79.47	87.36
18	18.51	28.55	37.70	46.41	54.84	63.07	71.14	79.12	86.97
19	18.41	28.42	37.54	46.21	54.61	62.80	70.87	78.80	86.64
20	18.32	28.30	37.39	46.04	54.41	62.58	70.60	78.52	86.33
25	17.99	27.85	36.83	45.38	53.55	61.74	69.55	77.48	85.20
30	17.79	27.56	36.47	44.96	53.16	61.18	69.05	76.81	84.48

TABLE VII (Continued)

$\alpha = 0.01 \qquad p = 4$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
5	45.48	67.95	88.52	108.33	127.50	146.51	165.13	183.54	201.82
6	38.41	58.52	77.05	94.81	112.11	129.07	145.79	162.32	178.70
7	34.80	53.66	71.05	87.72	103.95	119.86	135.53	151.06	166.42
8	32.58	50.66	67.33	83.30	98.86	114.12	129.14	143.99	158.73
9	31.08	48.61	64.78	80.27	95.36	110.17	124.74	139.17	153.43
10	30.00	47.12	62.92	78.07	92.92	107.28	121.54	135.60	149.57
11	29.18	45.99	61.50	76.38	90.57	105.07	119.07	132.89	146.62
12	28.54	45.10	60.39	75.06	89.33	103.34	117.13	130.77	144.28
13	28.03	44.38	59.48	73.98	88.09	101.92	115.56	129.03	142.39
14	27.60	43.79	58.74	73.09	87.05	100.75	114.26	127.60	140.81
15	27.24	43.29	58.12	72.34	86.19	99.78	113.16	126.40	139.49
16	26.94	42.86	57.59	71.71	85.45	98.94	112.24	125.35	138.36
17	26.68	42.50	57.13	71.17	84.82	98.23	111.42	124.46	137.41
18	26.46	42.18	56.72	70.68	84.26	97.59	110.71	123.70	136.56
19	26.26	41.90	56.38	70.26	83.79	97.04	110.11	123.02	135.81
20	26.08	41.65	56.06	69.89	83.36	96.55	109.56	122.42	135.13
25	25.44	40.74	54.92	68.52	81.77	94.75	107.55	120.19	132.73
30	25.03	40.17	54.19	67.64	80.75	93.60	106.26	118.78	131.15

$\alpha = 0.01 \qquad p = 5$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
6	63.59	96.63	127.33	156.94	185.34	214.26	242.31	270.07	297.64
7	53.60	83.24	110.79	137.35	163.28	188.83	214.03	238.93	263.64
8	48.35	76.11	101.91	126.76	151.07	174.97	198.58	221.92	245.07
9	45.09	71.61	96.28	120.07	143.30	166.13	188.68	211.02	233.21
10	42.86	68.51	92.37	115.39	137.88	159.97	181.82	203.43	224.84
11	41.23	66.22	89.50	111.93	133.85	155.44	176.71	197.81	218.68
12	39.99	64.47	87.29	109.29	130.76	151.90	172.78	193.43	213.96
13	39.00	63.08	85.53	107.18	128.32	149.13	169.68	190.01	210.18
14	38.21	61.95	84.10	105.46	126.31	146.86	167.08	187.15	207.06
15	37.55	61.01	82.90	104.02	124.66	144.93	164.97	184.83	204.48
16	37.00	60.23	81.90	102.82	123.26	143.34	163.17	182.85	202.34
17	36.52	59.55	81.04	101.78	122.04	141.98	161.57	181.15	200.45
18	36.12	58.97	80.30	100.88	120.99	140.77	160.31	179.65	198.86
19	35.77	58.46	79.65	100.10	120.08	139.76	159.14	178.35	197.42
20	35.46	58.01	79.09	99.41	119.29	138.82	158.11	177.22	196.18
25	34.33	56.38	77.00	96.91	116.36	135.49	154.38	173.09	191.64
30	33.62	55.37	75.70	95.34	114.52	133.40	152.02	170.48	188.82

TABLE VII (Continued)

$\alpha = 0.01 \qquad p = 6$

n_o	q=2	q=3	q=4	q=5	q=6	q=7	q=8	q=9	q=10
7	84.51	130.23	173.00	214.41	254.35	294.86	334.36	373.45	412.35
8	71.34	112.45	150.89	188.07	224.50	260.43	295.88	331.06	365.98
9	64.27	102.74	138.71	173.53	207.62	241.23	274.41	307.34	340.08
10	59.81	96.51	130.87	164.10	196.59	228.79	260.53	291.97	323.20
11	56.71	92.15	125.34	157.47	188.32	219.93	250.64	280.98	311.19
12	54.43	88.92	121.24	152.51	183.14	213.36	243.13	272.79	302.24
13	52.68	86.41	118.04	148.67	178.64	208.24	237.44	266.46	295.22
14	51.29	84.41	115.48	145.56	175.06	204.06	232.85	261.33	289.58
15	50.16	82.77	113.38	143.04	172.11	200.75	229.06	257.12	284.98
16	49.22	81.41	111.65	140.92	169.61	197.92	225.90	253.59	281.09
17	48.41	80.26	110.16	139.13	167.53	195.49	223.20	250.62	277.86
18	47.74	79.27	108.91	137.60	165.73	193.46	220.87	248.03	275.04
19	47.15	78.40	107.80	136.26	164.17	191.64	218.85	245.80	272.61
20	46.63	77.65	106.83	135.09	162.80	190.07	217.07	243.87	270.47
25	44.80	74.37	103.36	130.88	157.87	184.44	210.76	236.82	262.71
30	43.67	73.30	101.21	128.26	154.80	180.93	206.73	232.45	257.88

SECTION VII

LIKELIHOOD RATIO TEST STATISTIC FOR MULTIPLE HOMOGENEITY OF THE COVARIANCE MATRICES

In this section, we discuss Pearson type approximations to the distribution of the likelihood ratio statistic for testing the hypothesis H_5 when $p_i = p(i = 1,\ldots,q)$ and where H_5 was defined in Section II. This hypothesis is of interest in studying certain linear structures on the covariance matrices (see Krishnaiah and Lee [19]). The likelihood ratio statistic for testing H_5 is

$$V_5 = \frac{\prod_{i=1}^{q} \left| A_{ii}/n_i \right|^{n_i/2}}{\prod_{j=1}^{k} \left| \sum_{i=q_{j-1}^*+1}^{q_j^*} A_{ii}/n_j^* \right|^{n_j^*/2}} \tag{16}$$

where n_i and A_{ii} were defined in the preceding section, and

$$n_j^* = \sum_{i=q_{j-1}^*+1}^{q_j^*} n_i$$

The h-th moment of V_5 is given by

$$E(V_5^h) = \left[\frac{\prod_{\alpha=1}^{k} (n_\alpha^*)^{hn_\alpha^* p/2}}{\prod_{g=1}^{q} (n_\alpha)^{hn_g p/2}}\right] \prod_{i=1}^{p} \prod_{\alpha=1}^{k} \left\{ \prod_{g=q_{\alpha-1}^*+1}^{q_\alpha^*} \frac{\Gamma[\frac{1}{2}(n_g+hn_g+1-i)]}{\Gamma[\frac{1}{2}(n_g+1-i)]} \right.$$

$$\left. \cdot \frac{\Gamma[\frac{1}{2}(n_\alpha^*+1-i)]}{\Gamma[\frac{1}{2}(n_\alpha^*+hn_\alpha^*+1-i)]} \right\} \tag{17}$$

As in the preceding section, the distribution of $V_5^{1/b}$ is approximated with the Pearson's Type I distribution where b is a properly chosen integer. In Tables VIII and IX, we have $n_i = n_o$ and $q = kd$. In Table VIII, α_1 is the value of α obtained if we use Pearson type approximation whereas α_2 is the value of α obtained by using Box's asymptotic series up to order n^{-13}, where

$$P\left[V_5^* \leq c_5 \middle| H_5 \right] = (1-\alpha) \tag{18}$$

and $$V_5^* = -2 \log V_5.$$

Table VIII indicates that the accuracy of the Pearson type approximation is sufficient for practical purposes. Using the above approximation, we computed tables for the values of c_5 where $\alpha = 0.05, 0.01$, $k = 2, 3$, $M = 1(1)20(5)30$, $M = n_o - p$, $p = 1,2,3,4$. These values are given in Table IX.

TABLE VIII

COMPARISON OF THE PEARSON TYPE APPROXIMATION WITH THE

ASYMPTOTIC EXPRESSION FOR THE DISTRIBUTION OF V_5^*

			p=1			p=2			p=3			p=4		
n_o	q	k	c_5	α_1	α_2	c_5	α_1	α_2	c_5	α_1	α_2	c_5	α_1	α_2
10	6	2	8.20	0.05	0.0500	18.97	0.05	0.0500	34.63	0.05	0.0501	56.43	0.05	0.0503
10	6	2	11.89	0.01	0.0100	24.30	0.01	0.0100	41.77	0.01	0.0100	65.68	0.01	0.0101
10	6	3	9.90	0.05	0.0500	23.28	0.05	0.0500	42.79	0.05	0.0499	69.79	0.05	0.0498
10	6	3	13.85	0.01	0.0100	29.03	0.01	0.0100	50.53	0.01	0.0100	79.81	0.01	0.0099
20	6	2	8.01	0.05	0.0500	17.89	0.05	0.0500	31.45	0.05	0.0500	49.16	0.05	0.0500
20	6	2	11.62	0.01	0.0100	22.91	0.01	0.0100	37.93	0.01	0.0100	57.17	0.01	0.0100
20	6	3	9.70	0.05	0.0500	22.09	0.05	0.0500	39.30	0.05	0.0500	61.81	0.05	0.0499
20	6	3	13.57	0.01	0.0100	27.55	0.01	0.0100	46.39	0.01	0.0100	70.60	0.01	0.0100
30	6	2	7.94	0.05	0.0500	17.55	0.05	0.0500	30.54	0.05	0.0500	47.20	0.05	0.0500
30	6	2	11.53	0.01	0.0100	22.48	0.01	0.0100	36.81	0.01	0.0100	54.88	0.01	0.0100
30	6	3	9.63	0.05	0.0500	21.72	0.05	0.0500	38.28	0.05	0.0500	59.61	0.05	0.0500
30	6	3	13.47	0.01	0.0100	27.09	0.01	0.0100	45.17	0.01	0.0100	68.09	0.01	0.0100

TABLE IX

PERCENTAGE POINTS OF V_5^*

	$p = 1$, $\alpha = 0.05$				$p = 1$, $\alpha = 0.01$			
	$k = 2$		$k = 3$		$k = 2$		$k = 3$	
M	$q = 4$	$q = 6$	$q = 6$	$q = 9$	$q = 4$	$q = 6$	$q = 6$	$q = 9$
1	7.23	9.42	11.26	14.96	10.90	13.44	15.55	19.73
2	6.89	8.99	10.76	14.29	10.48	12.92	14.95	18.95
3	6.69	8.72	10.48	13.91	10.22	12.59	14.60	18.49
4	6.56	8.56	10.29	13.66	10.05	12.38	14.36	18.19
5	6.47	8.44	10.17	13.49	9.92	12.22	14.20	17.95
6	6.41	8.35	10.07	13.37	9.83	12.11	14.08	17.83
7	6.36	8.29	10.00	13.28	9.76	12.02	13.98	17.70
8	6.32	8.24	9.95	13.20	9.70	11.95	13.91	17.61
9	6.29	8.20	9.90	13.14	9.65	11.89	13.85	17.54
10	6.26	8.16	9.87	13.10	9.61	11.84	13.80	17.47
11	6.24	8.14	9.84	13.05	9.58	11.80	13.76	17.42
12	6.22	8.11	9.81	13.02	9.55	11.77	13.72	17.37
13	6.20	8.09	9.79	12.99	9.53	11.74	13.69	17.33
14	6.19	8.07	9.77	12.96	9.51	11.71	13.66	17.30
15	6.18	8.06	9.75	12.94	9.49	11.69	13.64	17.27
16	6.17	8.04	9.73	12.92	9.47	11.67	13.62	17.25
17	6.16	8.03	9.72	12.90	9.46	11.65	13.60	17.22
18	6.15	8.02	9.71	12.89	9.45	11.64	13.58	17.20
19	6.14	8.01	9.70	12.87	9.44	11.62	13.57	17.18
20	6.13	8.00	9.69	12.86	9.43	11.61	13.55	17.16
22	6.12	7.98	9.67	12.84	9.41	11.59	13.53	17.13
24	6.11	7.97	9.66	12.82	9.39	11.57	13.51	17.11
26	6.10	7.96	9.64	12.80	9.38	11.55	13.49	17.09
28	6.09	7.95	9.63	12.79	9.37	11.54	13.48	17.07
30	6.08	7.94	9.62	12.78	9.36	11.53	13.47	17.05

TABLE IX (Continued)

	p = 2, α = 0.05				p = 2, α = 0.01			
	k = 2		k = 3		k = 2		k = 3	
M	q = 4	q = 6	q = 6	q = 9	q = 4	q = 6	q = 6	q = 9
1	13.64	26.38	31.22	42.86	26.19	33.76	38.95	51.69
2	17.24	23.17	27.80	38.16	23.01	29.66	34.69	46.03
3	16.06	21.58	26.11	35.84	21.44	27.63	32.56	43.21
4	15.36	20.64	25.09	34.44	20.50	26.43	31.29	41.53
5	14.89	20.01	24.41	33.51	19.88	25.63	30.44	40.40
6	14.56	19.57	23.92	32.05	19.44	25.05	29.84	39.60
7	14.31	19.23	23.56	32.35	19.11	24.63	29.38	39.00
8	14.12	18.97	23.28	31.96	18.85	24.30	29.03	38.53
9	13.97	18.77	23.05	31.66	18.65	24.03	28.75	38.16
10	13.84	18.60	22.87	31.40	18.48	23.81	28.52	37.85
11	13.73	18.46	22.72	31.19	18.34	23.63	28.32	37.60
12	13.65	18.34	22.59	31.01	18.22	23.48	28.16	37.39
13	13.57	18.24	22.48	30.86	18.12	23.35	28.02	37.20
14	13.50	18.15	22.38	30.73	18.03	23.24	27.90	37.04
15	13.45	18.07	22.29	30.61	17.95	23.14	27.80	36.90
16	13.40	18.00	22.22	30.51	17.89	23.05	27.71	36.78
17	13.35	17.94	22.15	30.42	17.83	22.97	27.62	36.67
18	13.31	17.89	22.09	30.33	17.77	22.91	27.55	36.57
19	13.28	17.84	22.04	30.26	17.73	22.84	27.48	36.48
20	13.24	17.79	21.99	30.20	17.68	22.79	27.42	36.40
22	13.19	17.72	21.91	30.08	17.61	22.69	27.32	36.26
24	13.14	17.66	21.84	29.98	17.54	22.61	27.23	36.15
26	13.10	17.60	21.78	29.90	17.49	22.54	27.15	36.05
28	13.06	17.55	21.72	29.83	17.44	22.48	27.09	35.96
30	13.03	17.51	21.68	29.77	17.40	22.43	27.03	35.89

TABLE IX (Continued)

	p = 3, α = 0.05				p = 3, α = 0.01			
	k = 2		k = 3		k = 2		k = 3	
M	q = 4	q = 6	q = 6	q = 9	q = 4	q = 6	q = 6	q = 9
1	37.36	51.25	60.49	84.63	46.88	62.09	71.79	97.60
2	32.02	43.95	52.80	73.90	40.07	53.13	62.50	85.09
3	29.32	40.25	48.87	68.42	36.65	48.61	57.78	78.71
4	27.70	38.02	46.47	65.06	34.58	45.88	54.90	74.83
5	26.60	36.52	44.84	62.79	33.20	44.07	52.97	72.21
6	25.82	35.44	43.67	61.16	32.22	42.76	51.58	70.32
7	25.23	34.63	42.79	59.93	31.47	41.77	50.53	68.89
8	24.77	34.00	42.10	58.96	30.90	41.01	49.70	67.77
9	24.40	33.49	41.55	58.18	30.43	40.40	49.05	66.88
10	24.09	33.08	41.09	57.55	30.05	39.89	48.50	66.14
11	23.84	32.73	40.71	57.01	29.74	39.47	48.05	65.52
12	23.62	32.44	40.39	56.56	29.47	39.11	47.67	65.00
13	23.44	32.19	40.10	56.16	29.24	38.81	47.34	64.55
14	23.28	31.96	39.86	55.82	29.04	38.54	47.05	64.17
15	23.14	31.76	39.65	55.53	28.86	38.31	46.80	63.83
16	23.02	31.60	39.46	55.27	28.71	38.11	46.58	63.51
17	22.91	31.45	39.30	55.03	28.57	37.93	46.39	63.25
18	22.81	31.32	39.15	54.83	28.44	37.76	46.21	63.01
19	22.72	31.20	39.01	54.63	28.33	37.61	46.05	62.80
20	22.64	31.09	38.89	54.47	28.23	37.48	45.90	62.59
22	22.50	30.90	38.68	54.17	28.06	37.25	45.65	62.25
24	22.39	30.74	38.50	53.91	27.92	37.06	45.44	61.97
26	22.28	30.60	38.35	53.71	27.79	36.89	45.26	61.73
28	22.20	30.48	38.22	53.52	27.68	36.75	45.10	61.52
30	22.12	30.38	38.10	53.35	27.53	36.62	44.97	61.32

TABLE IX (Continued)

	$p = 4$ $\alpha = 0.05$				$p = 4$ $\alpha = 0.01$			
	$k = 2$		$k = 3$		$k = 2$		$k = 3$	
M	$q = 4$	$q = 6$	$q = 6$	$q = 9$	$q = 4$	$q = 6$	$q = 6$	$q = 9$
1	50.49	64.13	99.17	140.36	73.06	98.57	114.18	157.80
2	51.49	71.57	86.14	122.00	61.91	83.68	98.79	136.76
3	45.80	65.17	79.22	112.27	56.16	75.97	90.72	125.72
4	43.30	61.15	74.92	106.21	52.63	71.23	85.73	118.83
5	41.93	58.41	71.95	102.04	50.24	68.00	82.29	114.11
6	40.50	56.43	69.79	98.97	48.51	65.68	79.81	110.66
7	39.41	54.32	68.15	96.63	47.19	63.91	77.91	108.04
8	38.55	53.73	66.85	94.81	46.16	62.52	76.41	105.98
9	37.88	52.78	65.80	93.31	45.33	61.39	75.20	104.30
10	37.31	51.99	64.94	92.08	44.65	60.47	74.20	102.93
11	36.83	51.33	64.21	91.06	44.08	59.70	73.36	101.77
12	36.43	50.70	63.58	90.16	43.59	59.03	72.65	100.79
13	36.08	50.28	63.04	89.42	43.17	58.47	72.03	99.94
14	35.78	49.86	62.58	88.75	42.80	57.98	71.49	99.20
15	35.51	49.49	62.17	88.16	42.48	57.55	71.02	98.53
16	35.28	49.16	61.81	87.66	42.21	57.17	70.60	97.95
17	35.07	48.87	61.48	87.20	41.95	56.82	70.23	97.44
18	34.88	48.61	61.19	86.78	41.73	56.52	69.90	96.99
19	34.71	48.37	60.92	86.40	41.52	56.25	69.60	96.56
20	34.55	48.15	60.68	86.06	41.33	55.99	69.32	96.17
22	34.29	47.78	60.26	85.47	41.01	55.56	68.85	95.53
24	34.06	47.47	59.92	84.97	40.74	55.19	68.45	94.96
26	33.87	47.20	59.61	84.55	40.51	54.88	68.09	94.48
28	33.70	46.97	59.35	84.19	40.31	54.61	67.80	94.07
30	33.55	46.76	59.13	83.87	40.14	54.37	67.54	93.70

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APPENDIX

Let W $(0 \leq W \leq 1)$ be a random variable whose moments are given by

$$E(W^h) = C\left[\prod_{j=1}^{b} y_j^{y_j} \Big/ \prod_{i=1}^{a} x_i^{x_i}\right] \frac{\prod_{i=1}^{a} \Gamma[x_i(1+h)+\xi_i]}{\prod_{j=1}^{b} \Gamma[y_j(1+h)+\eta_j]}$$

where C is chosen such that $E(W^0) = 1$ and $\sum_{i=1}^{a} X_i = \sum_{j=1}^{b} Y_j$. Box (1949) gave an asymptotic expression for the distribution of $-2\rho \log W$, where ρ $(0 \leq \rho \leq 1)$ is an arbitrary constant.

In this appendix we give asymptotic expression for the distribution of $-2\rho \log W$ with remaining term of order N^{-15} where ρ is chosen such that $\omega_1 = 0$, ω_r is given by

$$\omega_r = \frac{(-1)^{r+1}}{r(r+1)}\left\{\sum_{i=1}^{a} \frac{B_{r+1}(\beta_i+\xi_i)}{(\rho x_i)^r} - \sum_{j=1}^{b} \frac{B_{r+1}(\varepsilon_j+\eta_j)}{(\rho y_j)^r}\right\}$$

$\beta_i = (1-\rho)X_i$, $\varepsilon_j = (1-\rho)Y_j$ and $B_r(h)$ is the Bernoulli polynomial of degree r and order unity.

Let $g_\nu(z)$ be the probability density function (p.d.f.) of the χ^2 distribution with ν degrees of freedom and N be the sample size. The p.d.f. of $-2\rho \log W$ is given by

$$\ell(z) = \sum_{r=0}^{14} S_r(z) + O(N^{-15})$$

where

$$S_0(\zeta) = g_f(\zeta), \quad S_1(\zeta) = 0, \quad S_2(\zeta) = \omega_2[\,g_{f+4}(\zeta) - g_f(\zeta)\,]$$

$$S_3(\zeta) = \omega_3[\,g_{f+6}(\zeta) - g_f(\zeta)\,]$$

$$S_4(\zeta) = \omega_4[\,g_{f+8}(\zeta) - g_f(\zeta)\,] + \tfrac{1}{2}\omega_2^2[\,g_{f+8}(\zeta) - 2\,g_{f+4}(\zeta) + g_f(\zeta)\,]$$

$$S_5(\zeta) = \omega_5[\,g_{f+10}(\zeta) - g_f(\zeta)\,] + \omega_2\omega_3[\,g_{f+10}(\zeta) - g_{f+6}(\zeta) - g_{f+4}(\zeta) + g_f(\zeta)\,]$$

$$\begin{aligned} S_6(\zeta) = {} & \omega_6[\,g_{f+12}(\zeta) - g_f(\zeta)\,] + \omega_4\omega_2[\,g_{f+12}(\zeta) - g_{f+8}(\zeta) - g_{f+4}(\zeta) + g_f(\zeta)\,] \\ & + \omega_3^2[\,\tfrac{1}{2}g_{f+12}(\zeta) - g_{f+6}(\zeta) + \tfrac{1}{2}g_f(\zeta)\,] \\ & + \omega_2^3[\,\tfrac{1}{6}g_{f+12}(\zeta) - \tfrac{1}{2}g_{f+8}(\zeta) + \tfrac{1}{2}g_{f+4}(\zeta) - \tfrac{1}{6}g_f(\zeta)\,] \end{aligned}$$

$$\begin{aligned} S_7(\zeta) = {} & \omega_7[\,g_{f+14}(\zeta) - g_f(\zeta)\,] + \omega_5\omega_2[\,g_{f+14}(\zeta) - g_{f+10}(\zeta) - g_{f+4}(\zeta) + g_f(\zeta)\,] \\ & + \omega_4\omega_3[\,g_{f+14}(\zeta) - g_{f+8}(\zeta) - g_{f+6}(\zeta) + g_f(\zeta)\,] \\ & + \omega_3\omega_2^2[\,\tfrac{1}{2}g_{f+14}(\zeta) - g_{f+10}(\zeta) - \tfrac{1}{2}g_{f+8}(\zeta) + \tfrac{1}{2}g_{f+6}(\zeta) + g_{f+4}(\zeta) - \tfrac{1}{2}g_f(\zeta)\,] \end{aligned}$$

$$\begin{aligned} S_8(\zeta) = {} & \omega_8[\,g_{f+16}(\zeta) - g_f(\zeta)\,] + \omega_6\omega_2[\,g_{f+16}(\zeta) - g_{f+12}(\zeta) - g_{f+4}(\zeta) + g_f(\zeta)\,] \\ & + \omega_5\omega_3[\,g_{f+16}(\zeta) - g_{f+10}(\zeta) - g_{f+6}(\zeta) + g_f(\zeta)\,] \\ & + \omega_4^2[\,\tfrac{1}{2}g_{f+16}(\zeta) - g_{f+8}(\zeta) + \tfrac{1}{2}g_f(\zeta)\,] \\ & + \omega_4\omega_2^2[\,\tfrac{1}{2}g_{f+16}(\zeta) - g_{f+12}(\zeta) + g_{f+4}(\zeta) - \tfrac{1}{2}g_f(\zeta)\,] \\ & + \omega_3^2\omega_2[\,\tfrac{1}{2}g_{f+16}(\zeta) - \tfrac{1}{2}g_{f+12}(\zeta) - g_{f+10}(\zeta) + g_{f+6}(\zeta) \\ & \qquad + \tfrac{1}{2}g_{f+4}(\zeta) - \tfrac{1}{2}g_f(\zeta)\,] \\ & + \omega_2^4[\,\tfrac{1}{24}g_{f+16}(\zeta) - \tfrac{1}{6}g_{f+12}(\zeta) + \tfrac{1}{4}g_{f+8}(\zeta) - \tfrac{1}{6}g_{f+4}(\zeta) + \tfrac{1}{24}g_f(\zeta)\,] \end{aligned}$$

$$S_9(z) = \omega_9[g_{f+18}(z) - g_f(z)] + \omega_7\omega_2[g_{f+18}(z) - g_{f+14}(z) - g_{f+4}(z) + g_f(z)]$$

$$+\omega_6\omega_3[g_{f+18}(z) - g_{f+12}(z) - g_{f+6}(z) + g_f(z)]$$

$$+\omega_5\omega_4[g_{f+18}(z) - g_{f+10}(z) - g_{f+8}(z) + g_f(z)]$$

$$+\omega_5\omega_2^2[\tfrac{1}{2} g_{f+18}(z) - g_{f+14}(z) + \tfrac{1}{2} g_{f+10}(z) - \tfrac{1}{2} g_{f+8}(z) + g_{f+4}(z) - \tfrac{1}{2} g_f(z)]$$

$$+\omega_4\omega_3\omega_2[g_{f+18}(z) - g_{f+14}(z) - g_{f+12}(z) - g_{f+10}(z) + g_{f+8}(z) + g_{f+6}(z)$$

$$+ g_{f+4}(z) - g_f(z)]$$

$$+\omega_3^3[\tfrac{1}{6} g_{f+18}(z) - \tfrac{1}{2} g_{f+12}(z) + \tfrac{1}{2} g_{f+6}(z) - \tfrac{1}{6} g_f(z)]$$

$$+\omega_3\omega_2^3[\tfrac{1}{6} g_{f+18}(z) - \tfrac{1}{2} g_{f+14}(z) - \tfrac{1}{6} g_{f+12}(z) + \tfrac{1}{2} g_{f+10}(z)$$

$$+\tfrac{1}{2} g_{f+8}(z) - \tfrac{1}{6} g_{f+6}(z) - \tfrac{1}{2} g_{f+4}(z) + \tfrac{1}{6} g_f(z)]$$

$$S_{10}(z) = \omega_{10}[g_{f+20}(z) - g_f(z)] + \omega_8\omega_2[g_{f+20}(z) - g_{f+16}(z) - g_{f+4}(z) + g_f(z)]$$

$$+\omega_7\omega_3[g_{f+20}(z) - g_{f+14}(z) - g_{f+6}(z) + g_f(z)]$$

$$+\omega_6\omega_4[g_{f+20}(z) - g_{f+12}(z) - g_{f+8}(z) + g_f(z)]$$

$$+\omega_6\omega_2^2[\tfrac{1}{2} g_{f+20}(z) - g_{f+16}(z) + \tfrac{1}{2} g_{f+12}(z) - \tfrac{1}{2} g_{f+8}(z)$$

$$+ g_{f+4}(z) - \tfrac{1}{2} g_f(z)]$$

$$+\omega_5^2[\tfrac{1}{2} g_{f+20}(z) - g_{f+10}(z) + \tfrac{1}{2} g_f(z)]$$

$$+\omega_5\omega_3\omega_2[g_{f+20}(z) - g_{f+16}(z) - g_{f+14}(z) + g_{f+6}(z) + g_{f+4}(z) - g_f(z)]$$

$$+\omega_4^2\omega_2[\tfrac{1}{2} g_{f+20}(z) - \tfrac{1}{2} g_{f+16}(z) - g_{f+12}(z) + g_{f+8}(z) + \tfrac{1}{2} g_{f+4}(z) - \tfrac{1}{2} g_f(z)]$$

$$+\ \omega_4\omega_3^2\left[\tfrac{1}{2}g_{f+20}(z)-g_{f+14}(z)-\tfrac{1}{2}g_{f+12}(z)+\tfrac{1}{2}g_{f+8}(z)+g_{f+6}(z)-\tfrac{1}{2}g_f(z)\right]$$

$$+\ \omega_3^2\omega_2^2\left[\tfrac{1}{4}g_{f+20}(z)-\tfrac{1}{2}g_{f+16}(z)-\tfrac{1}{2}g_{f+14}(z)+\tfrac{1}{4}g_{f+12}(z)+g_{f+10}(z)\right.$$

$$\left.+\tfrac{1}{4}g_{f+8}(z)-\tfrac{1}{2}g_{f+6}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{4}g_f(z)\right]$$

$$+\ \omega_2^3\omega_4\left[\tfrac{1}{6}g_{f+20}(z)-\tfrac{1}{2}g_{f+16}(z)+\tfrac{1}{3}g_{f+12}(z)+\tfrac{1}{3}g_{f+8}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{6}g_f(z)\right]$$

$$+\ \omega_2^5\left[\tfrac{1}{120}g_{f+20}(z)-\tfrac{1}{24}g_{f+16}(z)+\tfrac{1}{12}g_{f+12}(z)-\tfrac{1}{12}g_{f+8}(z)\right.$$

$$\left.+\tfrac{1}{24}g_{f+4}(z)-\tfrac{1}{120}g_f(z)\right]$$

$$S_{11}(z)=\omega_{11}\left[g_{f+22}(z)-g_f(z)\right]+\omega_9\omega_2\left[g_{f+22}(z)-g_{f+18}(z)-g_{f+4}(z)+g_f(z)\right]$$

$$+\ \omega_8\omega_3\left[g_{f+22}(z)-g_{f+16}(z)-g_{f+6}(z)+g_f(z)\right]$$

$$+\ \omega_7\omega_2^2\left[\tfrac{1}{2}g_{f+22}(z)-g_{f+18}(z)+\tfrac{1}{2}g_{f+14}(z)-\tfrac{1}{2}g_{f+8}(z)+g_{f+4}(z)-\tfrac{1}{2}g_f(z)\right]$$

$$+\ \omega_7\omega_4\left[g_{f+22}(z)-g_{f+14}(z)-g_{f+8}(z)+g_f(z)\right]$$

$$+\ \omega_6\omega_5\left[g_{f+22}(z)-g_{f+12}(z)-g_{f+10}(z)+g_f(z)\right]$$

$$+\ \omega_6\omega_3\omega_2\left[g_{f+22}(z)-g_{f+18}(z)-g_{f+16}(z)+g_{f+12}(z)-g_{f+10}(z)\right.$$

$$\left.+g_{f+6}(z)+g_{f+4}(z)-g_f(z)\right]$$

$$+\ \omega_5\omega_4\omega_2\left[g_{f+22}(z)-g_{f+18}(z)-g_{f+14}(z)-g_{f+12}(z)+g_{f+10}(z)+g_{f+8}(z)+g_{f+4}(z)\right.$$

$$\left.-g_f(z)\right]$$

$$+\ \omega_5\omega_3^2\left[\tfrac{1}{2}g_{f+22}(z)-g_{f+16}(z)-\tfrac{1}{2}g_{f+12}(z)+\tfrac{1}{2}g_{f+10}(z)\right.$$

$$\left.+g_{f+6}(z)-\tfrac{1}{2}g_f(z)\right]$$

$$+\omega_5\omega_2^3\Big[\tfrac{1}{6}g_{f+22}(z)-\tfrac{1}{2}g_{f+18}(z)+\tfrac{1}{2}g_{f+14}(z)-\tfrac{1}{6}g_{f+12}(z)$$

$$-\tfrac{1}{6}g_{f+10}(z)+\tfrac{1}{2}g_{f+8}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{6}g_f(z)\Big]$$

$$+\omega_4^2\omega_3\Big[\tfrac{1}{2}g_{f+22}(z)-\tfrac{1}{2}g_{f+16}(z)-g_{f+14}(z)+g_{f+8}(z)+\tfrac{1}{2}g_{f+6}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_4\omega_3\omega_2^2\Big[\tfrac{1}{2}g_{f+22}(z)-g_{f+18}(z)-\tfrac{1}{2}g_{f+16}(z)+g_{f+12}(z)+g_{f+10}(z)-\tfrac{1}{2}g_{f+6}(z)$$

$$-g_{f+4}(z)+\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_3^3\omega_2\Big[\tfrac{1}{6}g_{f+22}(z)-\tfrac{1}{6}g_{f+18}(z)-\tfrac{1}{2}g_{f+16}(z)+\tfrac{1}{2}g_{f+12}(z)$$

$$+\tfrac{1}{2}g_{f+10}(z)-\tfrac{1}{2}g_{f+6}(z)-\tfrac{1}{6}g_{f+4}(z)+\tfrac{1}{6}g_f(z)\Big]$$

$$+\omega_3\omega_2^4\Big[\tfrac{1}{24}g_{f+22}(z)-\tfrac{1}{6}g_{f+18}(z)-\tfrac{1}{24}g_{f+16}(z)+\tfrac{1}{4}g_{f+14}(z)$$

$$+\tfrac{1}{6}g_{f+12}(z)-\tfrac{1}{6}g_{f+10}(z)-\tfrac{1}{4}g_{f+8}(z)+\tfrac{1}{24}g_{f+6}(z)$$

$$+\tfrac{1}{6}g_{f+4}(z)-\tfrac{1}{24}g_f(z)\Big]$$

$$S_{12}(z)=\omega_{12}\big[g_{f+24}(z)-g_f(z)\big]+\omega_{10}\omega_2\big[g_{f+24}(z)-g_{f+20}(z)-g_{f+4}(z)+g_f(z)\big]$$

$$+\omega_9\omega_3\big[g_{f+24}(z)-g_{f+18}(z)-g_{f+6}(z)+g_f(z)\big]+\omega_8\omega_4\big[g_{f+24}(z)-g_{f+16}(z)-g_{f+8}(z)+g_f(z)\big]$$

$$+\omega_8\omega_2^2\Big[\tfrac{1}{2}g_{f+24}(z)-g_{f+20}(z)+\tfrac{1}{2}g_{f+16}(z)-\tfrac{1}{2}g_{f+8}(z)+g_{f+4}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_7\omega_5\big[g_{f+24}(z)-g_{f+14}(z)-g_{f+10}(z)+g_f(z)\big]$$

$$+\omega_7\omega_3\omega_2\big[g_{f+24}(z)-g_{f+20}(z)-g_{f+18}(z)+g_{f+14}(z)-g_{f+10}(z)+g_{f+6}(z)$$

$$+g_{f+4}(z)-g_f(z)\big]$$

$$+ \omega_6^2 \left[\tfrac{1}{2} g_{f+24}(z) - g_{f+12}(z) + \tfrac{1}{2} g_f(z) \right]$$

$$+ \omega_6 \omega_4 \omega_2 \left[g_{f+24}(z) - g_{f+20}(z) - g_{f+16}(z) + g_{f+8}(z) + g_{f+4}(z) - g_f(z) \right]$$

$$+ \omega_6 \omega_2^3 \left[\tfrac{1}{2} g_{f+24}(z) - \tfrac{1}{2} g_{f+20}(z) + \tfrac{1}{2} g_{f+16}(z) - \tfrac{1}{3} g_{f+12}(z) + \tfrac{1}{2} g_{f+8}(z) \right.$$

$$\left. - \tfrac{1}{2} g_{f+4}(z) + \tfrac{1}{6} g_f(z) \right]$$

$$+ \omega_6 \omega_3^2 \left[\tfrac{1}{2} g_{f+24}(z) - g_{f+18}(z) + g_{f+6}(z) - \tfrac{1}{2} g_f(z) \right]$$

$$+ \omega_5^2 \omega_2 \left[\tfrac{1}{2} g_{f+24}(z) - \tfrac{1}{2} g_{f+20}(z) - g_{f+14}(z) + g_{f+10}(z) + \tfrac{1}{2} g_{f+4}(z) - \tfrac{1}{2} g_f(z) \right]$$

$$+ \omega_5 \omega_4 \omega_3 \left[g_{f+24}(z) - g_{f+18}(z) - g_{f+16}(z) - g_{f+14}(z) + g_{f+10}(z) + g_{f+8}(z) \right.$$

$$\left. + g_{f+6}(z) - g_f(z) \right]$$

$$+ \omega_5 \omega_3 \omega_2^2 \left[\tfrac{1}{2} g_{f+24}(z) - g_{f+20}(z) - \tfrac{1}{2} g_{f+18}(z) + \tfrac{1}{2} g_{f+12}(z) + \tfrac{1}{2} g_{f+14}(z) \right.$$

$$\left. + \tfrac{1}{2} g_{f+10}(z) + \tfrac{1}{2} g_{f+8}(z) - \tfrac{1}{2} g_{f+6}(z) - g_{f+4}(z) + \tfrac{1}{2} g_f(z) \right]$$

$$+ \omega_4^3 \left[\tfrac{1}{6} g_{f+24}(z) - \tfrac{1}{2} g_{f+16}(z) + \tfrac{1}{2} g_{f+8}(z) - \tfrac{1}{6} g_f(z) \right]$$

$$+ \omega_4^2 \omega_2^2 \left[\tfrac{1}{4} g_{f+24}(z) - \tfrac{1}{2} g_{f+20}(z) - \tfrac{1}{4} g_{f+16}(z) + g_{f+12}(z) - \tfrac{1}{4} g_{f+8}(z) \right.$$

$$\left. - \tfrac{1}{2} g_{f+4}(z) + \tfrac{1}{4} g_f(z) \right]$$

$$+ \omega_4 \omega_3^2 \omega_2 \left[\tfrac{1}{2} g_{f+24}(z) - \tfrac{1}{2} g_{f+20}(z) - g_{f+18}(z) - \tfrac{1}{2} g_{f+16}(z) + g_{f+14}(z) \right.$$

$$\left. + g_{f+12}(z) + g_{f+10}(z) - \tfrac{1}{2} g_{f+8}(z) - g_{f+6}(z) - \tfrac{1}{2} g_{f+4}(z) + \tfrac{1}{2} g_f(z) \right]$$

$$+\omega_4\omega_2^4\Big[\tfrac{1}{24}g_{f+24}(z)-\tfrac{1}{6}g_{f+20}(z)+\tfrac{5}{24}g_{f+16}(z)-\tfrac{5}{24}g_{f+8}(z)+\tfrac{1}{6}g_{f+4}(z)$$

$$-\tfrac{1}{24}g_f(z)\Big]$$

$$+\omega_3^4\Big[\tfrac{1}{24}g_{f+24}(z)-\tfrac{1}{6}g_{f+18}(z)+\tfrac{1}{4}g_{f+12}(z)-\tfrac{1}{6}g_{f+6}(z)+\tfrac{1}{24}g_f(z)\Big]$$

$$+\omega_3^2\omega_2^3\Big[\tfrac{1}{12}g_{f+24}(z)-\tfrac{1}{4}g_{f+20}(z)-\tfrac{1}{6}g_{f+18}(z)+\tfrac{1}{4}g_{f+16}(z)$$

$$+\tfrac{1}{2}g_{f+14}(z)-\tfrac{1}{2}g_{f+10}(z)-\tfrac{1}{4}g_{f+8}(z)+\tfrac{1}{6}g_{f+6}(z)+\tfrac{1}{4}g_{f+4}(z)-\tfrac{1}{12}g_f(z)\Big]$$

$$+\omega_2^6\Big[\tfrac{1}{720}g_{f+24}(z)-\tfrac{1}{120}g_{f+20}(z)+\tfrac{1}{48}g_{f+16}(z)-\tfrac{1}{36}g_{f+12}(z)$$

$$+\tfrac{1}{48}g_{f+8}(z)-\tfrac{1}{120}g_{f+4}(z)+\tfrac{1}{720}g_f(z)\Big]$$

$$S_3(z)=\omega_{13}\big[g_{f+26}(z)-g_f(z)\big]+\omega_{11}\omega_2\big[g_{f+26}(z)-g_{f+22}(z)-g_{f+4}(z)+g_f(z)\big]$$

$$+\omega_{10}\omega_3\big[g_{f+26}(z)-g_{f+20}(z)-g_{f+6}(z)+g_f(z)\big]$$

$$+\omega_9\omega_4\big[g_{f+26}(z)-g_{f+18}(z)-g_{f+8}(z)+g_f(z)\big]$$

$$+\omega_9\omega_2^2\Big[\tfrac{1}{2}g_{f+26}(z)-g_{f+22}(z)+\tfrac{1}{2}g_{f+18}(z)-\tfrac{1}{2}g_{f+8}(z)+g_{f+4}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_8\omega_5\big[g_{f+26}(z)-g_{f+16}(z)-g_{f+10}(z)+g_f(z)\big]$$

$$+\omega_8\omega_3\omega_2\big[g_{f+26}(z)-g_{f+22}(z)-g_{f+20}(z)+g_{f+16}(z)-g_{f+10}(z)+g_{f+6}(z)+g_{f+4}(z)-g_f(z)\big]$$

$$+\omega_7\omega_6\big[g_{f+26}(z)-g_{f+14}(z)-g_{f+12}(z)+g_f(z)\big]$$

$$+\,\omega_7\omega_4\omega_2\Big[g_{f+26}(z)-g_{f+22}(z)-g_{f+18}(z)+g_{f+14}(z)-g_{f+12}(z)+g_{f+8}(z)+g_{f+4}(z)-g_f(z)\Big]$$

$$+\,\omega_7\omega_3^2\Big[\tfrac{1}{2}g_{f+26}(z)-g_{f+22}(z)+\tfrac{1}{2}g_{f+16}(z)-\tfrac{1}{2}g_{f+12}(z)+g_{f+6}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\,\omega_7\omega_2^3\Big[\tfrac{1}{6}g_{f+26}(z)-\tfrac{1}{2}g_{f+22}(z)+\tfrac{1}{2}g_{f+18}(z)-\tfrac{1}{6}g_{f+14}(z)-\tfrac{1}{6}g_{f+12}(z)$$

$$+\tfrac{1}{2}g_{f+8}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{6}g_f(z)\Big]$$

$$+\,\omega_6\omega_5\omega_2\Big[g_{f+26}(z)-g_{f+22}(z)-g_{f+16}(z)-g_{f+14}(z)+g_{f+12}(z)+g_{f+10}(z)+g_{f+4}(z)-g_f(z)\Big]$$

$$+\,\omega_6\omega_4\omega_3\Big[g_{f+26}(z)-g_{f+20}(z)-g_{f+18}(z)-g_{f+14}(z)+g_{f+12}(z)+g_{f+8}(z)+g_{f+6}(z)-g_f(z)\Big]$$

$$+\,\omega_6\omega_3\omega_2^2\Big[\tfrac{1}{2}g_{f+26}(z)-g_{f+22}(z)-\tfrac{1}{2}g_{f+20}(z)+\tfrac{1}{2}g_{f+18}(z)+g_{f+16}(z)$$

$$-\tfrac{1}{2}g_{f+14}(z)-\tfrac{1}{2}g_{f+12}(z)+g_{f+10}(z)+\tfrac{1}{2}g_{f+8}(z)-\tfrac{1}{2}g_{f+6}(z)$$

$$-\,g_{f+4}(z)+\tfrac{1}{2}g_f(z)\Big]$$

$$+\,\omega_5\omega_4^2\Big[\tfrac{1}{2}g_{f+26}(z)-g_{f+18}(z)-\tfrac{1}{2}g_{f+16}(z)+\tfrac{1}{2}g_{f+10}(z)+g_{f+8}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\,\omega_5\omega_4\omega_2^2\Big[\tfrac{1}{2}g_{f+26}(z)-g_{f+22}(z)-\tfrac{1}{2}g_{f+16}(z)+g_{f+14}(z)+g_{f+12}(z)-\tfrac{1}{2}g_{f+10}(z)$$

$$-\,g_{f+4}(z)+\tfrac{1}{2}g_f(z)\Big]$$

$$+\,\omega_5\omega_3^2\omega_2\Big[\tfrac{1}{2}g_{f+26}(z)-\tfrac{1}{2}g_{f+22}(z)-g_{f+20}(z)+\tfrac{1}{2}g_{f+16}(z)+\tfrac{1}{2}g_{f+14}(z)$$

$$+\tfrac{1}{2}g_{f+12}(z)+\tfrac{1}{2}g_{f+10}(z)-g_{f+6}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_5^2\omega_3\Big[\tfrac{1}{2}g_{f+26}(z)-\tfrac{1}{2}g_{f+20}(z)-g_{f+16}(z)+g_{f+10}(z)+\tfrac{1}{2}g_{f+6}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_5\omega_2^4\Big[\tfrac{1}{24}g_{f+26}(z)-\tfrac{1}{6}g_{f+22}(z)+\tfrac{1}{4}g_{f+18}(z)-\tfrac{1}{24}g_{f+16}(z)-\tfrac{1}{6}g_{f+14}(z)$$

$$+\tfrac{1}{6}g_{f+12}(z)+\tfrac{1}{24}g_{f+10}(z)-\tfrac{1}{4}g_{f+8}(z)+\tfrac{1}{6}g_{f+4}(z)-\tfrac{1}{24}g_f(z)\Big]$$

$$+\omega_4\omega_3^3\Big[\tfrac{1}{6}g_{f+26}(z)-\tfrac{1}{2}g_{f+20}(z)-\tfrac{1}{6}g_{f+18}(z)+\tfrac{1}{2}g_{f+14}(z)+\tfrac{1}{2}g_{f+12}(z)$$

$$-\tfrac{1}{6}g_{f+8}(z)-\tfrac{1}{2}g_{f+6}(z)+\tfrac{1}{6}g_f(z)\Big]$$

$$+\omega_4\omega_3\omega_2^3\Big[\tfrac{1}{6}g_{f+26}(z)-\tfrac{1}{2}g_{f+22}(z)-\tfrac{1}{6}g_{f+20}(z)+\tfrac{1}{3}g_{f+18}(z)+\tfrac{1}{2}g_{f+16}(z)$$

$$+\tfrac{1}{3}g_{f+14}(z)-\tfrac{1}{3}g_{f+12}(z)-\tfrac{1}{2}g_{f+10}(z)-\tfrac{1}{3}g_{f+8}(z)+\tfrac{1}{6}g_{f+6}(z)+\tfrac{1}{2}g_{f+4}(z)$$

$$-\tfrac{1}{6}g_f(z)\Big]$$

$$+\omega_4^2\omega_3\omega_2\Big[\tfrac{1}{2}g_{f+26}(z)-\tfrac{1}{2}g_{f+22}(z)-\tfrac{1}{2}g_{f+20}(z)-g_{f+18}(z)+\tfrac{1}{2}g_{f+16}(z)+g_{f+14}(z)$$

$$+g_{f+12}(z)+\tfrac{1}{2}g_{f+10}(z)-g_{f+8}(z)-\tfrac{1}{2}g_{f+6}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_3^3\omega_2^2\Big[\tfrac{1}{12}g_{f+26}(z)-\tfrac{1}{6}g_{f+22}(z)-\tfrac{1}{4}g_{f+20}(z)+\tfrac{1}{12}g_{f+18}(z)+\tfrac{1}{2}g_{f+16}(z)$$

$$+\tfrac{1}{4}g_{f+14}(z)-\tfrac{1}{4}g_{f+12}(z)-\tfrac{1}{2}g_{f+10}(z)-\tfrac{1}{12}g_{f+8}(z)+\tfrac{1}{4}g_{f+6}(z)$$

$$+\tfrac{1}{6}g_{f+4}(z)-\tfrac{1}{12}g_f(z)\Big]$$

$$+\omega_3\omega_2^5\Big[\tfrac{1}{120}g_{f+26}(z)-\tfrac{1}{24}g_{f+22}(z)-\tfrac{1}{12}g_{f+20}(z)+\tfrac{1}{12}g_{f+18}(z)$$

$$+\tfrac{1}{24}g_{f+16}(z)-\tfrac{1}{12}g_{f+14}(z)-\tfrac{1}{12}g_{f+12}(z)+\tfrac{1}{24}g_{f+10}(z)+\tfrac{1}{12}g_{f+8}(z)$$

$$-\tfrac{1}{120}g_{f+6}(z)-\tfrac{1}{24}g_{f+4}(z)+\tfrac{1}{120}g_f(z)\Big]$$

$$S_{14}(z)=\omega_{14}[g_{f+28}(z)-g_f(z)]+\omega_{12}\omega_2\big[g_{f+28}(z)-g_{f+24}(z)-g_{f+4}(z)+g_f(z)\big]$$

$$+\omega_{11}\omega_3\big[g_{f+28}(z)-g_{f+22}(z)-g_{f+6}(z)+g_f(z)\big]$$

$$+\omega_{10}\omega_4\big[g_{f+28}(z)-g_{f+20}(z)-g_{f+8}(z)+g_f(z)\big]$$

$$+\omega_{10}\omega_2^2\Big[\tfrac{1}{2}g_{f+28}(z)-g_{f+24}(z)+\tfrac{1}{2}g_{f+20}(z)-\tfrac{1}{2}g_{f+8}(z)+g_{f+4}(z)$$

$$-\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_9\omega_5\big[g_{f+28}(z)-g_{f+18}(z)-g_{f+10}(z)+g_f(z)\big]$$

$$+\omega_9\omega_3\omega_2\big[g_{f+28}(z)-g_{f+24}(z)-g_{f+22}(z)+g_{f+18}(z)-g_{f+10}(z)+g_{f+6}(z)$$

$$+g_{f+4}(z)-g_f(z)\big]$$

$$+\omega_8\omega_6\big[g_{f+28}(z)-g_{f+16}(z)-g_{f+12}(z)+g_f(z)\big]$$

$$+\omega_8\omega_3^2\Big[\tfrac{1}{2}g_{f+28}(z)-g_{f+22}(z)+\tfrac{1}{2}g_{f+16}(z)-\tfrac{1}{2}g_{f+12}(z)+g_{f+6}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\omega_8\omega_4\omega_2\big[g_{f+28}(z)-g_{f+24}(z)-g_{f+20}(z)+g_{f+16}(z)-g_{f+12}(z)+g_{f+8}(z)+g_{f+4}(z)-g_f(z)\big]$$

$$+\omega_8\omega_2^3\left[\frac{1}{6}g_{f+28}(z) - \frac{1}{2}g_{f+24}(z) + \frac{1}{2}g_{f+20}(z) - \frac{1}{6}g_{f+16}(z) - \frac{1}{6}g_{f+12}(z) + \frac{1}{2}g_{f+8}(z)\right.$$

$$\left. - \frac{1}{2}g_{f+4}(z) + \frac{1}{6}g_f(z)\right]$$

$$+\omega_7^2\left[\frac{1}{2}g_{f+28}(z) - g_{f+14}(z) + \frac{1}{2}g_f(z)\right]$$

$$+\omega_7\omega_5\omega_2\left[g_{f+28}(z) - g_{f+24}(z) - g_{f+18}(z) + g_{f+10}(z) + g_{f+4}(z) - g_f(z)\right]$$

$$+\omega_7\omega_4\omega_3\left[g_{f+28}(z) - g_{f+22}(z) - g_{f+20}(z) + g_{f+8}(z) + g_{f+6}(z) - g_f(z)\right]$$

$$+\omega_7\omega_3\omega_2^2\left[\frac{1}{2}g_{f+28}(z) - g_{f+24}(z) - \frac{1}{2}g_{f+22}(z) + \frac{1}{2}g_{f+20}(z) + g_{f+18}(z)\right.$$

$$\left. - g_{f+14}(z) + g_{f+10}(z) + \frac{1}{2}g_{f+8}(z) - \frac{1}{2}g_{f+6}(z) - g_{f+4}(z) + \frac{1}{2}g_f(z)\right]$$

$$+\omega_6^2\omega_2\left[\frac{1}{2}g_{f+28}(z) - \frac{1}{2}g_{f+24}(z) - g_{f+16}(z) + g_{f+12}(z) + \frac{1}{2}g_{f+4}(z) - \frac{1}{2}g_f(z)\right]$$

$$+\omega_6\omega_5\omega_3\left[g_{f+28}(z) - g_{f+22}(z) - g_{f+18}(z) - g_{f+16}(z) + g_{f+12}(z) + g_{f+10}(z) + g_{f+6}(z) - g_f(z)\right]$$

$$+\omega_6\omega_4^2\left[\frac{1}{2}g_{f+28}(z) - g_{f+20}(z) - \frac{1}{2}g_{f+16}(z) + \frac{1}{2}g_{f+12}(z) + g_{f+8}(z) - \frac{1}{2}g_f(z)\right]$$

$$+\omega_6\omega_4\omega_2^2\left[\frac{1}{2}g_{f+28}(z) - g_{f+24}(z) + \frac{1}{2}g_{f+16}(z) + \frac{1}{2}g_{f+12}(z) - g_{f+4}(z) + \frac{1}{2}g_f(z)\right]$$

$$+\omega_6\omega_3^2\omega_2\left[\frac{1}{2}g_{f+28}(z) - \frac{1}{2}g_{f+24}(z) - g_{f+22}(z) + g_{f+18}(z) + g_{f+10}(z) - g_{f+6}(z)\right.$$

$$\left. - \frac{1}{2}g_{f+4}(z) + \frac{1}{2}g_f(z)\right]$$

$$+\omega_6\omega_2^4\left[\frac{1}{24}g_{f+28}(z) - \frac{1}{6}g_{f+24}(z) + \frac{1}{4}g_{f+20}(z) - \frac{5}{24}g_{f+16}(z) + \frac{5}{24}g_{f+12}(z)\right.$$

$$\left. - \frac{1}{4}g_{f+8}(z) + \frac{1}{6}g_{f+4}(z) - \frac{1}{24}g_f(z)\right]$$

$$+\,\omega_5^2\omega_4\Big[\tfrac{1}{2}g_{f+28}(z)-\tfrac{1}{2}g_{f+24}(z)-g_{f+18}(z)+g_{f+10}(z)+\tfrac{1}{2}g_{f+8}(z)-\tfrac{1}{2}g_f(z)\Big]$$

$$+\,\omega_5\omega_3^3\Big[\tfrac{1}{6}g_{f+28}(z)-\tfrac{1}{2}g_{f+22}(z)-\tfrac{1}{6}g_{f+18}(z)+\tfrac{1}{2}g_{f+16}(z)+\tfrac{1}{2}g_{f+12}(z)$$

$$-\tfrac{1}{6}g_{f+10}(z)-\tfrac{1}{2}g_{f+6}(z)+\tfrac{1}{6}g_f(z)\Big]$$

$$+\,\omega_5^2\omega_2^2\Big[\tfrac{1}{4}g_{f+28}(z)-\tfrac{1}{2}g_{f+24}(z)+\tfrac{1}{4}g_{f+20}(z)-\tfrac{1}{2}g_{f+18}(z)+g_{f+14}(z)-\tfrac{1}{2}g_{f+10}(z)$$

$$+\tfrac{1}{4}g_{f+8}(z)-\tfrac{1}{2}g_{f+4}(z)+\tfrac{1}{4}g_f(z)\Big]$$

$$+\,\omega_5\omega_4\omega_3\omega_2\Big[g_{f+28}(z)-g_{f+24}(z)-g_{f+22}(z)-g_{f+20}(z)+g_{f+16}(z)$$

$$+2g_{f+14}(z)+g_{f+12}(z)-g_{f+8}(z)-g_{f+6}(z)-g_{f+4}(z)+g_f(z)\Big]$$

$$+\,\omega_5\omega_3\omega_2^3\Big[\tfrac{1}{6}g_{f+28}(z)-\tfrac{1}{2}g_{f+24}(z)-\tfrac{1}{6}g_{f+22}(z)+\tfrac{1}{2}g_{f+20}(z)$$

$$+\tfrac{1}{3}g_{f+18}(z)-\tfrac{1}{6}g_{f+16}(z)+\tfrac{1}{6}g_{f+12}(z)-\tfrac{1}{3}g_{f+10}(z)$$

$$-\tfrac{1}{2}g_{f+8}(z)+\tfrac{1}{6}g_{f+6}(z)+\tfrac{1}{2}g_{f+4}(z)-\tfrac{1}{6}g_f(z)\Big]$$

$$+\,\omega_4^3\omega_2\Big[\tfrac{1}{6}g_{f+28}(z)-\tfrac{1}{6}g_{f+24}(z)-\tfrac{1}{2}g_{f+20}(z)+\tfrac{1}{2}g_{f+16}(z)+\tfrac{1}{2}g_{f+12}(z)$$

$$-\tfrac{1}{2}g_{f+8}(z)-\tfrac{1}{6}g_{f+4}(z)+\tfrac{1}{6}g_f(z)\Big]$$

$$+\,\omega_3^4\omega_2\Big[\tfrac{1}{24}g_{f+28}(z)-\tfrac{1}{24}g_{f+24}(z)-\tfrac{1}{6}g_{f+22}(z)+\tfrac{1}{6}g_{f+18}(z)$$

$$+\tfrac{1}{4}g_{f+16}(z)-\tfrac{1}{4}g_{f+12}(z)-\tfrac{1}{6}g_{f+10}(z)+\tfrac{1}{6}g_{f+6}(z)$$

$$+\tfrac{1}{24}g_{f+4}(z)-\tfrac{1}{24}g_f(z)\Big]$$

$$+\omega_2^7\Big[\tfrac{1}{5040}g_{f+28}(z)-\tfrac{1}{720}g_{f+24}(z)+\tfrac{1}{240}g_{f+20}(z)$$

$$-\tfrac{1}{144}g_{f+16}(z)+\tfrac{1}{144}g_{f+12}(z)-\tfrac{1}{240}g_{f+8}(z)+\tfrac{1}{720}g_{f+4}(z)$$

$$-\tfrac{1}{5040}g_f(z)\Big]$$

$$+\omega_3^2\omega_2^4\Big[\tfrac{1}{48}g_{f+28}(z)-\tfrac{1}{12}g_{f+24}(z)-\tfrac{1}{24}g_{f+22}(z)+\tfrac{1}{8}g_{f+20}(z)$$

$$+\tfrac{1}{6}g_{f+18}(z)-\tfrac{1}{16}g_{f+16}(z)-\tfrac{1}{4}g_{f+14}(z)-\tfrac{1}{16}g_{f+12}(z)$$

$$+\tfrac{1}{6}g_{f+10}(z)+\tfrac{1}{8}g_{f+8}(z)-\tfrac{1}{24}g_{f+6}(z)-\tfrac{1}{12}g_{f+4}(z)+\tfrac{1}{48}g_f(z)\Big]$$

$$+\omega_4^2\omega_3^2\Big[\tfrac{1}{4}g_{f+28}(z)-\tfrac{1}{2}g_{f+22}(z)-\tfrac{1}{2}g_{f+20}(z)+\tfrac{1}{4}g_{f+16}(z)+g_{f+14}(z)$$

$$+\tfrac{1}{4}g_{f+12}(z)-\tfrac{1}{2}g_{f+8}(z)-\tfrac{1}{2}g_{f+6}(z)+\tfrac{1}{4}g_f(z)\Big]$$

$$+\omega_4^2\omega_2^3\Big[\tfrac{1}{12}g_{f+28}(z)-\tfrac{1}{4}g_{f+24}(z)+\tfrac{1}{12}g_{f+20}(z)+\tfrac{5}{12}g_{f+16}(z)$$

$$-\tfrac{5}{12}g_{f+12}(z)-\tfrac{1}{12}g_{f+8}(z)+\tfrac{1}{4}g_{f+4}(z)-\tfrac{1}{12}g_f(z)\Big]$$

$$+\omega_4\omega_3^2\omega_2^2\Big[\tfrac{1}{4}g_{f+28}(z)-\tfrac{1}{2}g_{f+24}(z)-\tfrac{1}{2}g_{f+22}(z)+g_{f+18}(z)$$

$$+\tfrac{3}{4}g_{f+16}(z)-\tfrac{3}{4}g_{f+12}(z)-g_{f+10}(z)+\tfrac{1}{2}g_{f+6}(z)$$

$$+\tfrac{1}{2}g_{f+4}(z)-\tfrac{1}{4}g_f(z)\Big]$$

$$+\omega_4\omega_2^5\Big[\tfrac{1}{120}g_{f+28}(z)-\tfrac{1}{24}g_{f+24}(z)+\tfrac{3}{40}g_{f+20}(z)-\tfrac{1}{24}g_{f+16}(z)-\tfrac{1}{24}g_{f+12}(z)$$

$$+\tfrac{3}{40}g_{f+8}(z)-\tfrac{1}{24}g_{f+4}(z)+\tfrac{1}{120}g_f(z)\Big]$$

and $f=-2\left\{\sum_{i=1}^{a} z_i-\sum_{j=1}^{b}\eta_j-\tfrac{1}{2}(a-b)\right\}$

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